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Commande périodique de bioréacteurs multi-spécifiques en vue de l'optimisation de leurs rendements

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Résumé : Les contributions de ce travail sont présentées en deux parties. Nous considérons d'abord un problème de contrôle optimal périodique régi par un système dynamique scalaire, linéaire par rapport à la variable de contrôle satisfaisant une contrainte intégrale. Nous présentons des conditions suffisantes permettant de déduire l'existence d'un sur-rendement qui consiste à améliorer le critère à l'équilibre associé un contrôle constant $\bar{u}$ en considérant un contrôle périodique u avec une valeur moyenne égale à $\bar{u}$. Nous utilisons le Principe de Maximum de Pontryagin pour conclure la synthèse optimale périodique satisfaisant la contrainte intégrale. Nous montrons le rôle important des hypothèses de la convexité et de la monotonie des données. Ces résultats sont appliqués au modèle du chémostat où l'objectif est d'améliorer la qualité de l'eau en considérant un débit périodique sous contrainte intégrale sur la quantité totale d'eau à traiter. Nous démontrons également une propriété de dualité permettant de considérer un deuxième problème dit dual, où l'on cherche un débit périodique qui maximise la quantité d'eau à traiter sur une période $[0, T]$ et pour laquelle, la valeur moyenne en substrat doit respecter un seuil. En se basant sur ces résultats, nous avons proposé un algorithme robuste permettant de distinguer entre deux types de cinétiques et qui combine entre des opérations stationnaires et périodiques. Dans un autre contexte, nous montrons comment garantir la résilience dans le modèle de chemostat en présence d'une espèce envahissante, dans un sens faible que nous définissons. Nous construisons une fonction débit qui varie au cours du temps et qui permet à l'espèce native de revenir au-dessus d'un seuil fixé, un nombre infini de fois, sans éradiquer l'espèce envahissante. Avec cette fonction, nous montrons que le temps passé par l'espèce native au-dessus du seuil est infini, on dit alors qu'elle est "faiblement résiliente". Nous prouvons ainsi qu'il existe une solution périodique unique du système associée à cette fonction faiblement résiliente et on conjecture que toute autre solution converge asymptotiquement vers cette solution périodique. Enfin, nous montrons que cela peut être réalisé avec un contrôle hybride qui ne nécessite pas une connaissance parfaite des caractéristiques de croissance des espèces.
Mots clés : contrôle optimal, contrainte intégrale, Principle de Maximum de Pontryagin, solutions périodiques, sur-rendement, solutions asymptotiquement périodiques, modèle du chémostat, faible résilience, contrôle hybride.

Abstract : The contributions of the thesis are twofold. We first consider a periodic optimal control problem governed by a one-dimensional system, linear with respect to the control variable and satisfying an integral constraint. We give sufficient conditions for over-yielding that consists in improving the criterion at steady state with a constant control $\bar{u}$ by considering a periodic control u with average value equal to $\bar{u}$. We use Pontryagin's Maximum Principle to provide the optimal synthesis of periodic strategies satisfying the integral constraint. Convexity and monotonicity assumptions are playing a crucial role. Theses results are applied to the chemostat model where the goal is to improve the averaged water quality using periodic removal rate under integral constraint on the total amount of water to be treated. We prove also a duality property allowing to consider a dual problem, which consists in improving the total quantity of treated water over a given time period $[0, T]$, compared to steady-state, by considering periodic operation under integral constraint on the water quality. Based on these results, we proposed a robust algorithm that distinguishes between two types of kinetics and combines stationary and periodic operations.

In another context, we show how resilience in the chemostat model in presence of a species invader can be guaranteed in a weak sense. We give a construction of a time varying removal rate allowing the resident species to come back above a fixed threshold, an infinite number of times, even though the invader can never be totally eradicated. With this control, we show that the time spent by the resident species above the threshold is of infinite measure, and thus the control is said to be "weakly resilient". We show that there exists an unique periodic solution of the system associated with such a time-varying removal rate and conjecture that any other trajectory converges asymptotically to this periodic solution. Finally, we show that this can be achieved by a hybrid feedback controller based on very few knowledge on the growth characteristics of the species.
Keywords : optimal control, integral constraint, Pontryagin Maximum Principle, periodic solutions, over-yielding, asymptotically periodic solutions, chemostat model, weak resilience, hybrid control.

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CHAPITRE 1

INTRODUCTION

Dans cette thèse, nous faisons l'étude mathématique de différents problèmes liés au contrôle périodique d'un système dynamique. Le premier consiste à caractériser le gain que peut apporter une loi de contrôle périodique non constante pour un critère d'optimisation. Le deuxième consiste à rechercher une loi de contrôle non constante qui permet aux solutions du système de restaurer certaines propriétés perdues avec une commande constante au cours du temps. Dans ce cas également, nous montrons que la loi de contrôle proposée amène les solutions du système à être asymptotiquement périodiques. Nous présentons dans l'introduction les résultats principaux des cinq chapitres dont est constituée la thèse avant de donner les contributions présentées sous forme d'articles dans les chapitres suivants.

1 Contexte scientifique

Tout d'abord et avant de détailler les résultats développés dans cette thèse, nous introduisons brièvement dans cette section quelques notions utiles pour comprendre le contenu des sections qui suivent.

1.1 Théories du contrôle et du contrôle optimal

En mathématiques, la théorie du contrôle vise à analyser les propriétés des systèmes dynamiques commandés, c'est à dire, les systèmes sur lesquels on peut agir à l'aide d'une commande. Cette définition recouvre naturellement de très nombreux champs d'applications: mécanique, économie, chimie, biologie, aéronautique, etc. De manière générale, un système de contrôle autonome s'écrit:

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)), \\ x(0) = x_0 \end{cases} \tag{1.1}$$

où $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ est la dynamique, $x(t)$ désigne l'état du système à l'instant $t \in [0, T]$ et $u(\cdot)$ la variable du contrôle (dite aussi commande) qui est une fonction mesurable bornée et à valeur dans un domaine quelconque $U \subset \mathbb{R}^m$.

En premier lieu, on se pose une question dite de contrôlabilité, qui se formule ainsi: étant donné un état $x_f \in \mathbb{R}^n$, existe-t-il un contrôle u tel que la trajectoire associée relie x_0 à x_f en un temps fini T? Après affirmation de la condition précédente, le deuxième problème consiste à trouver un contrôle reliant x_0 à x_f et qui de plus optimise un certain critère dit *coût* (par exemple: en dépensant le minimum d'énergie, en cherchant à le faire en un temps minimal, etc). C'est un problème de contrôle optimal. L'existence de trajectoires optimales dépend des propriétés du système et du coût [BCD08], [BP07], [Vin10] (régularité, convexité...).

La loi de contrôle peut être réalisée de deux manières fondamentalement différentes:

- en boucle ouverte: c'est à dire que u est une fonction du temps $t \mapsto u(t)$, valable pour une condition initiale précise,
- en boucle fermée (ou retour d'état dit *feedback* en anglais): le contrôle u dépend de l'état $x \mapsto u(x)$ valable pour tous les états initiaux possibles .

La mise en œuvre d'un contrôle en boucle ouverte est plus facile, puisque la seule information nécessaire est de mesurer le temps. Par contre, pour mettre en œuvre un contrôle en boucle fermée, il faut constamment mesurer l'état du système. Le contrôle par retour d'état présente toutefois des avantages puisqu'il est plus robuste en présence de perturbations qui ne peuvent pas être prévues à l'avance et s'applique pour différentes conditions initiales possibles.

Un autre pan de la théorie du contrôle est celui de la stabilisation: étant donné un état d'équilibre instable d'un système dynamique (pour un contrôle constant, souvent pris égal à 0), est-il possible de rendre celui-ci stable en appliquant un contrôle adéquat? En général, la stabilisation se fait à l'aide de lois de commande par retour d'état.

1.2 Bio-procédés: intérêt et contrôle

En ce qui concerne les bio-procédés, nous nous intéressons au chémostat qui est un dispositif expérimental développé en 1950 par Novick et Szilard [NS50a] et Monod [Mon50] simultanément. Il est utilisé pour contrôler et comprendre la croissance de populations de micro-organismes ([SW95], [HLRS17]) ainsi que leur adaptation évolutive. Il s'agit précisément d'un réacteur alimenté en continu en des ressources nutritives, appelées substrat et qui sont consommées par une ou plusieurs populations. Les nutriments résiduels et les micro-organismes sont soutirés du milieu de culture (enceinte réactionnelle) à la même vitesse que celle en entrée, ce qui permet le maintien de la culture, dans le réacteur, à un volume constant. Le contenu du chémostat est continuellement agité de façon à ce que le mélange reste parfaitement homogène. En plus, cette installation est équipée en général des dispositifs de régulation permettant de garder les paramètres environnementaux (notamment pH et température) constants. Trois modes de fonctionnement sont possibles dans un bioréacteur:

- En discontinu (batch en anglais): l'entrée et la sortie sont nulles ($D = 0$ dans (1.2)). Les micro-organismes ont une croissance exponentielle.
- En semi-continu (fed-batch en anglais): seule la sortie est nulle (volume non constant). C'est un mode de fonctionnement beaucoup utilisé en industrie pour contrôler la concentration du substrat.
- En continu: le débit de la sortie est égal au débit de l'entrée. Le volume reste constant dans le réservoir (D constant et non nul dans (1.2)).

Le chémostat fait partie du troisième type de fonctionnement (et c'est le mode étudié dans cette thèse). Il peut être schématisé comme dans la Fig. 1.1.

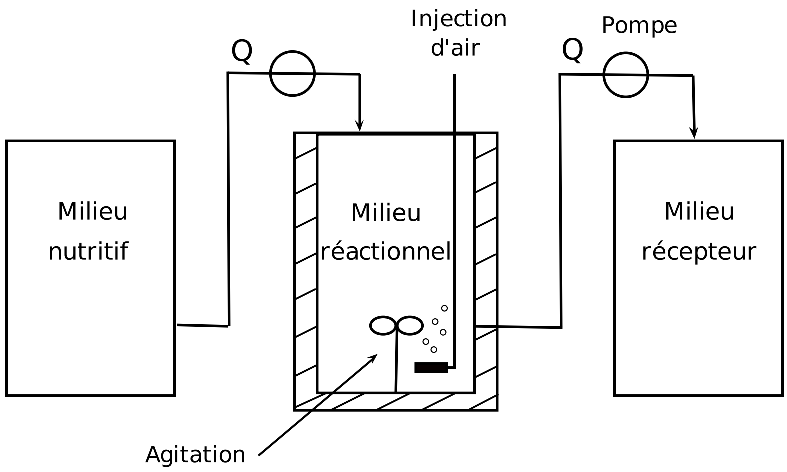


FIGURE 1.1 – *Schéma du chémostat.*

Chaque population dans le bioréacteur est caractérisée par un taux d'absorption de ressources et un taux de croissance (qui est souvent supposé proportionnel au taux d'absorption). A ce dispositif expérimental correspond un modèle mathématique qui décrit l'évolution et les interactions entres espèces et qui est obtenu en faisant le bilan de matière entre les instants t et $t + dt$. Considérons une enceinte à volume constant V et notons par $s(t)$, $x(t)$ les concentrations en substrat et en biomasse à l'instant t (mesurés en $[g/l]$), Q le débit d'alimentation et de soutirage (mesuré en $[l]/t$), s_{in} la concentration de ressources nutritionnelles en entrée et μ la fonction de croissance (appelée aussi « vitesse spécifique de croissance »). Nous présentons le modèle le plus simple, donné par un système d'équations différentielles ordinaires décrivant les évolutions des concentrations en micro-organismes (une seule espèce) et en substrat:

$$\left\{ \begin{array}{lcl} \dot{s} & = & D(s_{in} - s) - \frac{\mu(s)}{Y} x, \\ \dot{x} & = & \mu(s)x - Dx, \end{array} \right. \tag{1.2}$$

où $D := \frac{Q}{V}$ est appelé taux de dilution (mesuré en $1/t$), Y est appelé rendement de conversion, déterminé par les propriétés biologiques du système. Le taux de croissance $\mu(\cdot)$ peut dépendre des deux variables s et x, il s'appelle dans ce cas taux de croissance densité dépendant. Dans les deux cas, ces fonctions sont positives et nulles en $s = 0$. Plusieurs cinétiques (ou fonctions de croissance) sont utilisées dans la littérature. Nous citons dans la suite quelques exemples:

- Loi de Monod [Mon50] (identique à la fonction de Michaelis–Menten): elle est monotone et bornée, sa borne supérieur est noté μ_{max}. Elle est de la forme:
$$\mu(s) := \frac{\mu_{max} s}{K_s + s}, \ s \geq 0,$$
où $K_s > 0$ est la constante de demi-saturation. Cette loi permet de rendre compte des phénomènes de saturation et limitation.

- Loi de Haldane [And68] qui s'écrit:
$$\mu(s) := \frac{\bar{\mu}_{max} s}{K_s + s + s^2/K_I}, \ s \geq 0,$$
où $\bar{\mu}_{max}$ et K_I sont respectivement le taux de croissance maximal et la constante d'inhibition. Cette loi permet de rendre compte des phénomènes de saturation et inhibition.

- Loi de Hill [Mos58] qui est donnée par:
$$\mu(s) := \frac{\mu_{max} s^n}{K_s^n + s^n}, \ s \geq 0, \ n > 1,$$
et qui présente un faible effet Allee pour de faibles valeurs de s.

- Loi de Contois (voir [HLRS17]) qui est densité dépendant et donnée par:
$$\mu(s, x) := \frac{\mu_{max} s}{s + Kx}, \ s \geq 0, \ x \geq 0,$$
la constante de demi-saturation dans ce cas vaut Kx qui tend vers 0 lorsque x tend vers 0. Pour tout x, la fonction $s \mapsto \mu(s, x)$ est de type Monod.

Des résultats théoriques détaillés sur le comportement asymptotique de ce type de modèles avec différents types de fonctions de croissance sont présentés dans [HLRS17].
Le chémostat joue un rôle très important dans le domaine industriel. Il est largement utilisé dans le traitement des eaux usées par voie biologique

[LWN06] [RPB09] (épuration biologique ou bien dépollution biologique aérobie [1] et anaérobie [2]). Cette opération consiste à cultiver des micro-organismes sur des éléments nutritifs (par exemple le carbone, l'azote et le phosphore) présents dans la culture afin de réduire leurs concentrations. Un des buts intéressants de la digestion biologique anaérobie consiste à produire de l'énergie sous forme de biogaz (méthane). Ce dernier permet par la suite de produire de la chaleur qui pourrait être transformée en électricité. Une autre utilité du chémostat concerne la production de certains produits chimiques comme les protéines, les antibiotiques, etc.

Aujourd'hui, il est devenu possible de mesurer en ligne les variables du système (concentration en substrat ou en biomasse) grâce au développement des capteurs. Ce développement ainsi que le progrès de la modélisation mathématique ont ouvert la voie pour le contrôle des bio-procédés. Ce domaine a été largement développé pour servir au suivi des variables, à la stabilisation du système, à l'optimisation des performances, au rejet des perturbations et également pour ramener et maintenir les variables d'intérêt dans un ensemble de consignes désirées.

Comme nous l'avons mentionné précédemment, les bioréacteurs sont largement utilisés pour les traitement des eaux usées. Dans ce sens, la théorie du contrôle a été considérablement utilisée afin de chercher les stratégies optimales qui réduisent la concentration des éléments polluants, et cela en faisant passer le flux à travers un ou plusieurs bioréacteurs en série. Parmi les travaux existants dans ce sens, on cite [ZC14] où on montre que la concentration en substrat à l'équilibre peut être réduite si on répartit le volume total sur plusieurs bioréacteurs connectés en série. Dans [HFAB16], les auteurs traitent un problème de l'épuration de l'eau dans un réacteur biologique séquentiel (SBR [3] qui fonctionne en mode batch). Les SBR permettent un meilleur contrôle du procédé et donc une bonne qualité des eaux rejetées, par rapport aux réacteurs continus. Les auteurs dans [HFAB16] font l'étude de différents problèmes de contrôle optimal dont un consiste à ame-

1. C'est un traitement biologique durant lequel la matière organique présente dans les eaux résiduaires est assimilée par les micro-organismes, en présence d'oxygène.

2. Même processus précédent mais qui se fait sans présence d'oxygène.

3. En anglais Sequenting Batch Reactors, c'est un réacteur dans lequel les micro-organismes responsables de l'épuration sont maintenus en suspension et aérés. Il est alimenté en mode séquentiel et discontinu. Le processus de traitement dans un SBR suit au minimum quatre étapes: remplissage avec de l'eau usée, aération, décantation et puis évacuation de l'eau épurée

ner en temps minimal les concentrations en substrat en dessous de certains seuils prescrits, la variable de contrôle étant la concentration en oxygène. D'autres travaux se sont intéressés à la mise au point d'algorithmes de contrôle optimal en temps minimal des SBR [Mor99], [Maz06]. Ces stratégies permettent de minimiser la durée totale de la réaction.

1.3 Phénomènes périodiques: exemples pratiques

Un des problèmes intéressants en contrôle consiste à trouver des lois de commande u qui varient au cours du temps, pour lesquelles les solutions associées reviennent à l'instant initial après un instant T, c'est à dire $x(0) = x(T)$. L'intérêt pour ce type de problèmes est apparu dans les années 60 et a été appliqué aux problèmes industriels. Depuis lors, la même idée a été développée dans d'autres domaines d'application, tels que l'aéronautique [Gil75], [Spe73], [Spe76], le contrôle de l'énergie solaire [DK79], les sciences économiques [Fei92], [FN94], [MH97].

En génie chimique, les expérimentateurs ont observé les bénéfices potentiels du fonctionnement périodique des réacteurs chimiques sur la performance. Traditionnellement, les réacteurs chimiques sont conduits en régime continu, sous des conditions d'alimentation stationnaires (température, pression, concentration des réactants). L'objectif principal de ces travaux expérimentaux consiste à augmenter la performance de tels réacteurs (conversion d'un produit chimique à un autre, production de certains composants chimiques, etc). Deux questions principales liées à l'imposition du régime permanent périodique ont été posées:

- Peut-on améliorer un rendement obtenu dans des conditions stationnaires optimales, par la création d'un régime non-stationnaire périodique?
- Comment exploiter de manière optimale le potentiel d'un tel choix?

Le fonctionnement périodique des réacteurs chimiques a fait l'objet de plusieurs travaux de recherche comme [Lin66], [Fje69], [HL67], [BHL71] et aussi [Bai74]. Dans chacune de ces études, les auteurs ont mis en évidence l'effet positif du fonctionnement périodique sur la performance des réacteurs chimiques par rapport à un fonctionnement stationnaire et optimal. Dans la Section 2.1, nous expliquons brièvement les méthodes et les résultats principaux développés dans ces travaux.

En bio-procédés, la variation périodique des paramètres opératoires (D et s_{in}) dans des modèles du chémostat a intéressé beaucoup de chercheurs, au début des années 80. Le but principal du pilotage périodique d'un bioréacteur consiste essentiellement à amener les micro-organismes, qui sont en compétition, à coexister. En effet, lorsque plusieurs espèces sont présentes dans un chémostat, elles sont généralement en compétition pour une même ressource nécessaire à leur croissance. La théorie mathématique affirme que lorsque l'enceinte est alimentée avec un débit constant, une espèce va éliminer toutes les autres: c'est le « principe d'exclusion compétitive » ([SW95], [HLRS17]). Nous présentons dans la Section 2.4 les résultats théoriques liés à la compétition en chémostat. Le principe d'exclusion compétitive étant en général établi dans des conditions environnementales bien maîtrisées, dans [HHW77], on montre la coexistence pour un modèle de chémostat particulier (avec D et s_{in} constants) sous une condition non générique (égalité des seuils de croissance). Pour cela, les travaux sur la compétition en chémostat qui suivent se consacrent à la recherche d'autres situations permettant la coexistence. L'une des pistes envisagées consiste à créer un environnement variable: soit faire varier la concentration des nutriments s_{in} ou bien le taux de dilution D ou bien les deux à la fois. La variation périodique de s_{in} est naturelle d'un point de vue écologique car les niveaux de nutriments dans de nombreux écosystèmes [4] pourraient varier en fonction du temps (par exemple le cycle jour-nuit). Dans [HS83], [Hsu80], [Smi81], [SFA79], les auteurs ont montré que la coexistence des compétiteurs est en effet possible. Parallèlement, dans [SFA79], [BHW85], [LP95], [LP94], [NT05], on traite le cas de la variation périodique du taux de dilution D, dans un modèle de chémostat à plusieurs espèces et on montre que cette situation peut conduire à la coexistence. Dans tous ces exemples, des conditions d'existence et de stabilité des solutions périodiques ont été

4. Un écosystème comprend un milieu (biotope), des êtres vivants (biocénose) qui y vivent, s'y nourrissent et s'y reproduisent, ainsi que toutes les relations qui peuvent exister et se développer à l'intérieur de ce système.

établies à l'aide de techniques de bifurcation et de la théorie de Floquet qui entraînent la coexistence des espèces. Il faut souligner que les travaux énoncés ci–dessus peuvent être considérés comme des exemples de commande en boucle ouverte pour le modèle de compétition en chémostat.

2 Sur-rendement à l'aide d'un contrôle périodique: étude mathématique

Dans cette section, nous présentons dans un premier temps les principaux résultats mathématiques existants sur l'intérêt de la commande périodique. Ensuite, nous résumons la contribution des Chapitres 2 et 3 et enfin, nous abordons les résultats du Chapitre 4.

2.1 Historique

Dans les travaux de [Lin66], [Fje69], [HL67], [Bai74], [BHL71], les auteurs se sont intéressés principalement aux commandes périodiques ainsi qu'à leur influence sur la performance des réacteurs chimiques. A l'aide des techniques d'approximation numérique, ils sont arrivés à synthétiser des lois de commandes, de type bang-bang [5] en général. Ils utilisent principalement les méthodes itératives suivantes:

- Méthode de "Newton-Raphson" qui permet de donner une approximation d'un zéro d'une fonction. Cette méthode est utilisé dans ce contexte afin de calculer une trajectoire périodique qui correspond à une commande périodique donnée.
- A partir d'une commande périodique donnée et d'un critère de contrôle approprié, la méthode de "Hill-climbing" permet de corriger le processus périodique donné pour améliorer le coût. Cette amélioration peut être effectuée de manière itérative jusqu'à ce qu'aucune autre amélioration significative ne soit possible.

5. Une loi de commande $u : \mathbb{R} \to [u_{min}, u_{max}]$ est dite de type bang-bang si elle prend des valeurs dans l'ensemble $\{u_{min}, u_{max}\}$.

Ces études reposent sur le principe du maximum de Pontryagin (dit PMP, voir Annexe A) et l'analyse "relaxed steady-state" [BH71]. La commande stationnaire optimale et l'état qui lui correspond ont été utilisés comme référence pour l'évaluation de l'opération périodique optimale.

Dans [BFG73], les auteurs ont proposé un test permettant de déterminer si une opération périodique peut être avantageuse par rapport au fonctionnement à entrée constante, en fonction de la fréquence. Il s'agit d'un test appelé π-*test* qui a été revisité dans [BG80]. Le problème de contrôle optimal et périodique (OPC) a été introduit de la façon suivante (dans [BFG73] et sans contraintes pour simplifier):

(OPC): Soient $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ et $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^l$ deux fonctions de classe C^1, trouver parmi les fonctions $u(\cdot)$ appartenant à l'ensemble $L^\infty([0,T];\mathbb{R}^m)$ celles qui minimisent la fonction objectif J donnée par:

$$J_T(u) := \frac{1}{T}\int_0^T g(x(t),u(t))\,\mathrm{d}t,\ T > 0, \tag{1.3}$$

avec la dynamique et la condition de périodicité:

$$\begin{cases} \dot{x} = f(x,u), \\ x(0) = x(T). \end{cases} \tag{1.4}$$

Soit $(\bar{x},\bar{u})$ l'optimum dans la classe des solutions constantes de (1.4)-(1.3), dit aussi optimum du problème stationnaire (OSS) (*i.e.* $(\bar{x},\bar{u})$ minimise J sous la contrainte $f(\bar{x},\bar{u}) = 0$) et soit $\bar{J}$ son coût associé. La définition d'un contrôle *propre* et d'un contrôle *localement propre* qui a été présentée dans [BFG73] est la suivante.

Définition 2.1. *Un problème de contrôle optimal périodique (OPC) est dit propre s'il existe un $T > 0$ et un contrôle admissible $\hat{u}(\cdot)$ tel que $J(\hat{u}) < \bar{J}$.*
De même, (OPC) est dit localement propre s'il existe un $T > 0$ et une variation faible $\delta u(\cdot)$ tel que $J(\bar{u} + \delta u) < \bar{J}$.

Pour déduire le π-*test* (développé dans [BFG73]), une variation quadratique locale du critère J_T autour de la solution optimal à l'équilibre et des expressions des dérivées partielles du Hamiltonien ont été utilisées.

L'objectif de cette approche est d'évaluer s'il existe une fréquence d'un signal périodique (sous forme sinusoïdale en général) proche du régime stationnaire optimal $(\bar{x}, \bar{u})$ qui pourrait améliorer le coût. D'autres travaux (comme [BLM74], [WS87], [SE84], [Gil77]) ont également établi des conditions nécessaires et suffisantes d'optimalité, basées essentiellement sur le résultat de [BFG73]. Le critère π-*test* a été largement utilisé dans de nombreux problèmes, comme outil prédictif. A titre d'exemple, on peut citer [AL87], [Par98], [Par00], [Par03], [SY90], [SY91], [WOM81]. Par contre, ce critère présente quelques limitations, dans le sens que sa précision de calcul est réduite à de petites amplitudes. D'une manière générale, l'application du contrôle périodique dans le domaine industriel est restée considérée comme "trop avancée" et d'une complexité superflue [BC09] .

Une réflexion qui semble naturelle lorsque'on souhaite comparer la performance de deux régimes (ici système avec commande périodique ou stationnaire) consiste à imposer certaines contraintes sur la commande (des contraintes qui relient les deux structures). Cela permet alors de rendre cette comparaison réaliste. Toutefois, dans les articles que nous avons discutés précédemment, cette considération n'a pas été prise en compte. En bio-procédés, faire passer une même quantité de matière avec un débit périodique (qui est la variable du contrôle) ramène à imposer une contrainte intégrale (dite aussi isopérimétrique [6]) sur la variable de commande. Dans ce sens, on peut citer [SB80], [ZSMB17] et [BSMZ17] dans lesquels les auteurs ont proposé des lois de commandes périodiques, satisfaisant ce type de contraintes. Leur analyse est faite soit à l'aide de l'étude du domaine fréquentiel ou bien en appliquant le Principe du Maximum de Pontryagin, afin d'obtenir une amélioration des performances. Dans [ZSMB17], les auteurs ont posé deux problèmes d'optimisation périodique, sous contraintes isopérimétriques sur la commande:

- Maximiser la conversion d'un réactant en un produit souhaité.
- Maximiser la productivité d'un réacteur chimique.

Dans les deux problèmes, l'application du Principe du Maximum de Pon-

6. Soit $\bar{u} \in \mathbb{R}$. Une contrainte de type isopérimétrique signifie que l'on considère des contrôles $u : [0, T] \to \mathbb{R}$ mesurables vérifiant $\frac{1}{T}\int_0^T u(t)\mathrm{d}t = \bar{u}$.

tryagin qui donne des conditions nécessaires d'optimalité, a été déployée. Il faut noter que, dans l'étude du premier problème, les auteurs proposent des lois de commande de type bang-bang avec une seule commutation. Par contre, la recherche de la synthèse optimale globale n'a pas été faite. Pour le deuxième problème, le Principe du Maximum de Pontryagin est appliqué non pas pour le système initial mais plutôt pour le système linéarisé.

2.2 Résumé du Chapitre 2

Le second chapitre de cette thèse a été motivé principalement par un problème en bio-procédés. En effet, l'objectif de l'épuration biologique est de restaurer une "bonne" qualité d'eau à la fin du processus, en dégradant les principales substances polluantes. Cet objectif constitue en fait la principale motivation du développement de techniques d'optimisation des procédés de traitement et de dépollution. Cela pourrait être traduit en un problème de contrôle optimal dont le but consiste à déterminer des lois de commande pour maintenir la pollution de sortie, qui est la concentration en substrat, à un niveau bas.
Nous considérons le modèle de chémostat à une seule espèce (1.2) et nous supposons, sans perte de généralité que $Y = 1$. Pour une valeur adéquate $\bar{D}$ choisie dans (D_{min}, D_{max}) de manière à éviter l'extinction de la population de micro-organisme, nous notons par $\bar{s}$ le plus petit équilibre positif et stable du système (1.2) associé à $\bar{D}$. Alors, notre intérêt se résume à chercher l'avantage qu'apporte un débit un $D \in [D_{min}, D_{max}]$ variable (où D_{min} et D_{max} désignent respectivement le taux de dilution minimal et maximal) et qui génère des variables d'état (substrat et micro-organismes) périodiques, par rapport au pilotage stationnaire avec $\bar{D}$, sur la qualité de l'eau. Dans un premier temps, nous imposons une contrainte isopérimétrique sur la variable du contrôle de la forme $\frac{1}{T}\int_0^T D(t)\,\mathrm{d}t \geq \bar{D}$. D'un point de vue pratique, cette contrainte veut dire qu'on souhaite traiter une quantité au moins égale à $\bar{D}T$ sur une période $[0, T]$. La question qu'on se pose est la suivante:

Pb 1: Soit une quantité $\bar{D}T$ à traiter durant une période $[0, T]$, peut-on garantir une concentration moyenne en substrat, générée par un débit $D(\cdot)$ T-périodique, inférieure à $\bar{s}$, sous la contrainte $\frac{1}{T}\int_0^T D(t)\,\mathrm{d}t \geq \bar{D}$?

Dans un deuxième temps, on considère un problème un peu différent du premier puisqu'on impose une contrainte sur la concentration en substrat et pas sur le débit. Avec un débit non-constant $D \in [D_{min}, D_{max}]$, on impose que la moyenne de concentration de polluants en sortie ne dépasse pas $\bar{s}$ c'est à dire $\frac{1}{T}\int_0^T s(t)\,\mathrm{d}t \leq \bar{s}$. De même, on se pose la question suivante:

Pb 2: Soit un seuil $\bar{s}$ (équilibre associé à $\bar{D}$), peut-on garantir une quantité de polluants à traiter avec un débit $D(\cdot)$ T-périodique (donné par $s_{in}\frac{1}{T}\int_0^T Q(t)\,\mathrm{d}t$ avec $Q = DV$) supérieur à celle traitée avec le débit constant $\bar{D}$ (qui est $s_{in}\bar{Q}$ avec $\bar{Q} = \bar{D}V$), sous la contrainte $\frac{1}{T}\int_0^T s(t)\,\mathrm{d}t \leq \bar{s}$?

Par la suite, il s'agit de rechercher les stratégies appropriées qui minimisent la concentration de polluants en sortie (pour le premier problème) ou bien qui maximisent la quantité d'eau qui doit être traitée (pour le deuxième problème). Cela se traduit en deux problèmes de contrôle optimal que nous allons traiter dans la suite.

Ainsi, le deuxième chapitre de la thèse porte sur l'étude globale d'un problème d'optimisation périodique, sous contrainte intégrale sur la commande. Nous le présentons sous la forme:

$$(P_1): \begin{cases} \min J_T(u) := \frac{1}{T}\int_0^T \ell(x(t))\,\mathrm{d}t,\ T > 0 & (1.5)\\ \dot{x} = f(x) + u(t)g(x), & (1.6)\\ x(0) = x(T), & (1.7)\\ \frac{1}{T}\int_0^T u(t)\mathrm{d}t = \bar{u}. & (1.8) \end{cases}$$

Les fonctions ℓ, f et g sont supposées de classe C^1 et à valeurs dans $\mathbb{R}$ et u est une commande mesurable et à valeurs dans $[-1, 1]$. Il est important, d'un point de vue mathématique, d'introduire les hypothèses suivantes: il existe deux réels a, b tels que:

$$\begin{cases} g > 0 \text{ sur l'intervalle } (a, b)\\ f(a) - g(a) = 0,\ f(a) + g(a) = 0,\\ (f - g)(x) < 0,\ (f + g)(x) > 0,\ \forall x \in (a, b). \end{cases} \quad (1.9)$$

Cela implique d'abord que le domaine $I := (a, b)$ est invariant, c'est à dire

que toute trajectoire $x(\cdot)$, solution de (1.6), associé à $u \in [-1, 1]$ et issue d'une condition initiale prise dans I va rester dans I. Ainsi, imposer le signe de $\dot{x}(\cdot)$ avec $u \in \{-1, 1\}$ est important lors de la détermination et l'étude des trajectoires optimales, en termes de contrôlabilité sur I.
Nous supposons aussi qu'il existe un équilibre stable du système (1.6) dans I noté $\bar{x}$ associé à une commande constante $\bar{u} \in (-1, 1)$, c'est à dire $f(\bar{x}) + \bar{u}g(\bar{x}) = 0$. Il faut souligner que, dans notre cas, $(\bar{x}, \bar{u})$ n'est pas nécessairement solution du problème d'optimisation stationnaire (OSS) (qui consiste à minimiser la fonction objective J_T avec $f(\bar{x}) + \bar{u}g(\bar{x}) = 0$). Il est seulement demandé que $\bar{x}$ soit un équilibre localement stable du système (1.6), associé à la commande constante $\bar{u}$. Puisque nous nous intéressons aux solutions T-périodique de (1.6) sous (1.8), la contrainte (1.8) est alors équivalente à:

$$\frac{1}{T} \int_0^T \psi(x(t))\, \mathrm{d}t = \psi(\bar{x}), \tag{1.10}$$

avec $\psi : I \to [-1, 1]$ définie par:

$$\psi(x) := \frac{-f(x)}{g(x)},$$

avec $\psi(\bar{x}) = \bar{u}$. Dans un premier temps, nous montrons que (1.6)-(1.7)-(1.8) admet des solutions pour tout $T > 0$. Plus précisément, on cherche à savoir s'il existe au moins une variable de commande u qui varie dans l'intervalle $[0, T]$ vérifiant la contrainte (1.8) et qui génère des solutions non-constantes vérifiant $x(0) = x(T)$. Si c'est le cas, en répétant cette commande sur les intervalles $[iT, (i + 1)T]$, $i = 1, 2, \cdots$, on obtient des solutions T-périodiques. Il n'est pas nécessaire d'imposer la condition $u(t) = u(t + T)$ comme dans [AL87]. La variable de commande étant une fonction mesurable, elle peut être discontinue.
Afin de garantir l'existence des solutions T-périodiques de (1.6), nous introduisons une hypothèse sur la fonction ψ de la manière suivante:

$$(\psi(x) - \psi(\bar{x}))(x - \bar{x}) > 0, \ \forall x \in I. \tag{1.11}$$

Cette hypothèse garantie également la stabilité de $\bar{x}$, l'équilibre du système (1.6) associé à la commande $\bar{u} = \psi(\bar{x})$.

Nous proposons dans le Chapitre 2 une définition mathématique de *sur-rendement*, équivalente à celle du contrôle *propre*, dans notre contexte.

Définition 2.2. *Un sur-rendement existe pour le coût* (1.5) *s'il existe une paire* (x, u) *solution de* (1.6)-(1.7)-(1.8) *qui vérifie* $J_T(u) < J_T(\bar{u})$.

Dans l'étude théorique du problème de contrôle optimal (P_1), nous allons aborder trois points importants:

1. Trouver des conditions simples sur les données du problème (f, g et ℓ) permettant de conclure l'existence d'un sur-rendement, pour toute valeur de T.
2. Après avoir établi l'existence d'un sur-rendement, trouver la forme des solutions périodiques et optimales du problème (P_1).
3. Comment assouplir les conditions d'existence (du point 1) afin d'avoir un sur-rendement pour des valeurs restrictives de T?

A la fin, nous présentons quelques exemples applicatifs (notamment un qui traite l'épuration de l'eau en chémostat) avec des simulations numériques qui illustrent les résultats théoriques.

2.2.1 Conditions d'existence d'un sur-rendement

Pour répondre à la question d'existence d'un sur-rendement, nous utilisons une méthode qui repose sur un résultat important d'analyse mathématique: l'inégalité de Jensen[7]. C'est une propriété valable pour les fonctions convexes (ou bien concaves).

Pour illustrer cette idée, revenons au (Pb1) introduit précédemment avec contrainte d'égalité. Dans ce cas particulier et d'après la Définition 2.2, nous

7. Soient $I \subset \mathbb{R}$, $a, b \in \mathbb{R}$, avec $a < b$ et soient les fonctions $f \in C([a,b], I)$ et $\phi : I \to \mathbb{R}$ convexe. Alors

$$\phi\left(\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right) \leq \frac{1}{b-a}\int_a^b (\phi \circ f)(x)\,\mathrm{d}x.$$

L'inégalité est stricte ($<$) si la fonction ϕ est strictement convexe.

cherchons à avoir

$$\frac{1}{T}\int_0^T s(t)\,\mathrm{d}t < \bar{s},$$

sous la contrainte $\frac{1}{T}\int_0^T D(t)\,\mathrm{d}t = \bar{D}$. Remarquons que toute solution périodique de (1.2) vérifie nécessairement $s(t) + x(t) = s_{in}$ pour tout $t \geq 0$. Le système (1.2) peut être réduit en un système uni-dimensionnel qui s'écrit:

$$\dot{s} = (-\mu(s) + D)(s_{in} - s). \tag{1.12}$$

Un changement de variables simple nous permet d'écrire (1.12) sous la forme de (1.6), il suffit de poser $u := aD + b \in [-1, 1]$ avec:

$$a := \frac{2}{D_{max} - D_{min}}\ ,\ b := -\frac{D_{max} + D_{min}}{D_{max} - D_{min}}$$

et pour $s \in I$, on a:

$$\begin{aligned} f(s) &:= (-\mu(s) - b/a)(s_{in} - s),\\ g(s) &:= (s_{in} - s)/a,\\ \ell(s) &:= s,\\ \psi(s) &:= a\mu(s) - b. \end{aligned}$$

De plus, l'équation (1.10) devient:

$$\frac{1}{T}\int_0^T \mu(s(t))\,\mathrm{d}t = \mu(\bar{s}). \tag{1.13}$$

Supposons que le taux de croissance μ est de type Monod qui est une fonction strictement concave croissante sur $\mathbb{R}_+$. Si un sur-rendement existe alors d'après la monotonie de μ, nous obtenons:

$$\mu\left(\frac{1}{T}\int_0^T s(t)\,\mathrm{d}t\right) < \mu(\bar{s}). \tag{1.14}$$

Or, d'après l'inégalité de Jensen sur les fonctions strictement concaves, on a:

$$\mu\left(\frac{1}{T}\int_0^T s(t)\,\mathrm{d}t\right) > \frac{1}{T}\int_0^T \mu(s(t))\,\mathrm{d}t. \tag{1.15}$$

En rassemblant les inégalités (1.13)-(1.14)-(1.15), on obtient une contradiction. Dans le cas contraire, si la fonction μ est convexe croissante sur l'intervalle, un sur-rendement est vérifié.

Dans le cas général (problème (P_1)), nous proposons une hypothèse de la forme:

$$\begin{cases} \ell \text{ est croissante sur } I, \\ \gamma := \psi \circ \ell^{-1} \text{ est strictement convexe croissante sur } \ell(I), \end{cases} \tag{1.16}$$

qui permet de déduire le sur-rendement pour toute valeur de T. Il s'agit de la Proposition 2.1 du Chapitre 2.

Proposition 2.1. *Sous les hypothèses* (1.9) *et* (1.16), *un sur-rendement existe pour toute valeur de* $T > 0$.

2.2.2 Détermination des solutions optimales

En introduisant une variable d'état supplémentaire, l'étude du problème d'optimisation (P_1) se simplifie. En effet, nous proposons de réécrire la contrainte intégrale (1.8) sous forme d'équation différentielle. Le nouveau système dynamique à étudier est alors de dimension deux et s'écrit:

$$\begin{cases} \dot{x} = f(x) + u(t)g(x), \\ \dot{y} = u(t), \end{cases} \tag{1.17}$$

avec:

$$x(0) = x(T),\ y(0) = 0,\ y(T) = \bar{u}T. \tag{1.18}$$

On suppose, pour simplifier, que la condition initiale $x(0)$ est fixée à $\bar{x}$ (nous montrons que cela peut être imposé sans perte de généralité). Alors, le problème de contrôle optimal consiste en:

$$(P_2) : \inf_{u \in \mathcal{U}} \frac{1}{T} \int_0^T \ell(x(t))\,\mathrm{d}t \quad \text{t.q. } (x,y) \text{ satisfait (1.17)} - \text{(1.18)}.$$

Ici, la recherche des trajectoires optimales est effectuée sous les hypothèses d'existence d'un sur-rendement données par (1.16).

Les conditions d'optimalité pour un problème d'optimisation visent à en caractériser des solutions localement optimales, appelées extrémales. Les conditions nécessaires permettent de définir des propriétés qualitatives et quantitatives de ces solutions et de déduire analytiquement les solutions du problème d'optimisation. Le *Hamiltonien* associé au problème (P_2) s'écrit:

$$H = H(x, y, \lambda_x, \lambda_y, \lambda_0, u) = \lambda_0 \ell(x) + \lambda_x f(x) + u(\lambda_x g(x) + \lambda_y),$$

où $\lambda := (\lambda_x, \lambda_y)$ est l'état adjoint associé à la trajectoire et solution du système adjoint:

$$\begin{cases} \dot{\lambda}_x = -\lambda_0 \ell'(x) - \lambda_x (f'(x) + u(t) g'(x)), \\ \dot{\lambda}_y = 0, \end{cases}$$

pour presque tout $t \in [0, T]$ et λ_0 est une constante négative ou nulle. Le Principe de Maximum de Pontryagin nous donne la condition de maximisation suivante:

$$H(x(t), \lambda(t), \lambda_0, u(t)) \geq H(x(t), \lambda(t), \lambda_0, \omega),$$

pour presque tout $t \in [0, T]$ et tout $\omega \in [-1, 1]$. Dans notre cas d'étude, le Hamiltonien est linéaire en le contrôle, la fonction de commutation $\phi(\cdot)$ est alors:

$$t \mapsto \phi(t) := \lambda_x(t) g(x(t)) + \lambda_y.$$

Par des raisonnements sur le signe de λ_x et de la dérivée de $\phi(\cdot)$ et la périodicité des solutions $x(\cdot)$, nous avons montré qu'une extrémale est nécessairement normale[8]. Comme le système (1.17) est autonome, le Hamiltonien H est constant le long des trajectoires extrémales, ce qui nous a permis de déduire quelques propriétés de ces trajectoires. Nous montrons dans la Proposition 3.1 et la Remarque 3.2, le résultat suivant.

Proposition 2.2. *Sous les hypothèses* (1.9)-(1.16), *une extrémale vérifie nécessairement les propriétés suivantes:*

1. *Elle est de type bang-bang (le contrôle alterne entre* -1 *et* 1*) ayant au moins deux commutations (nous avons éliminé les arcs singuliers à l'optimalité).*

2. *Le nombre de commutation est pair.*

3. *Les temps de commutation correspondent exactement aux instants où les*

8. Une extrémale est dite normal si $\lambda_0 \neq 0$, voir Annexe A.

solutions $x(\cdot)$ changent de monotonie. A chaque temps de commutation $t_s \in (0, T)$, nous avons $x(t_s) \in \{x_M, x_m\}$ avec x_M, x_m sont respectivement le maximum et minimum de $x(\cdot)$ dans $(0, T)$.

Pour éliminer les arcs singulier à l'optimalité, il existe une autre méthode différente à celle utilisé dans le Chapitre 2 qui repose sur la condition de Legendre-Clebsch. Dans l'Annexe A, nous vérifions que cette condition n'est pas satisfaite le long de l'extrémale grâce aux hypothèses (1.16).

La proposition précédente nous permet de déduire une propriété importante concernant le coût associé aux extrémales du problème (P_2). En effet, nous verrons dans le Chapitre 2 qu'une extrémale ayant $2n$ ($n \in \mathbb{N}$, $n > 1$) commutations sur $(0, T)$ est nécessairement T/n-périodique puisque la commande alterne entre -1 et 1 et que sa trajectoire associée atteint en permanence, en chaque commutation, les mêmes valeurs x_m et x_M. De plus, la restriction de cette même extrémale sur l'intervalle $(0, T/n)$ admet exactement deux commutations. Alors, on peut vérifier que le coût associé à la première extrémale, qui a $2n$ commutations sur $(0, T)$, est identique au coût associé à sa restriction sur l'intervalle $(0, T/n)$.

Après avoir établi ce résultat fondamental, nous focalisons notre étude sur les trajectoires bang-bang à deux commutations seulement. La principale difficulté réside dans la preuve d'existence et d'unicité de telles trajectoires vérifiant à la fois la contrainte de périodicité et la contrainte (1.8), pour un T fixé. Pour cela, nous définissons des trajectoires $x_{BB}(\cdot)$ dans l'intervalle $[0, T]$ qui démarrent en $\bar{x}$ à $t = 0$ et qui sont associées à une commande bang-bang u_{BB} qui commute deux fois entre 1 et -1 en des instants t_1 et t_2 appartenant à $(0, T)$. De plus, nous posons $x_M := x_{BB}(t_1)$ et $x_m := x_{BB}(t_2)$. Par le biais de la fonction η définie sur I par:

$$\eta(x) := \frac{1}{f(x) + g(x)} - \frac{1}{f(x) - g(x)},$$

nous montrons que pour tout T fixé, une trajectoire $x_{BB}(\cdot)$ T-périodique associée à u_{BB} satisfaisant la contrainte (1.8), vérifie nécessairement (voir

Lemme 3.2):

$$\begin{cases} \displaystyle\int_{x_m}^{x_M} \eta(x)\,\mathrm{d}x = T, & (1.19)\\ \displaystyle\int_{x_m}^{x_M} \eta(x)\psi(x)\,\mathrm{d}x = \bar{u}T. & (1.20)\end{cases}$$

L'équation (1.19) résulte de $x_{BB}(0) = x_{BB}(T) = \bar{x}$ et (1.20) résulte du fait que u_{BB} vérifie (1.8).

En tenant compte que ces trajectoires particulières sont caractérisées par les paires (x_m, x_M), alors montrer l'existence et l'unicité de $x_{BB}(\cdot)$ et u_{BB} vérifiant respectivement la périodicité et la contrainte intégrale (1.8) revient à montrer l'existence et l'unicité de (x_m, x_M) vérifiant (1.19)-(1.20).
Pour se faire, nous effectuons une analyse des équations intégrales (1.19)-(1.20) pour tout T fixé. Cela se fait en deux parties. La première consiste à montrer que sous les hypothèse (1.9), pour chaque $\alpha \in I$ et $T > 0$, il existe un unique $\beta_T(\alpha)$ vérifiant:

$$\int_{\alpha}^{\beta_T(\alpha)} \eta(x)\,\mathrm{d}x = T,$$

et que la fonction $\beta : [a,b] \to [a,b]$ est croissante bijective. Alors, de l'équation (1.19), $x_M = \beta(x_m)$ est unique. La question qui se pose maintenant est de savoir si ces deux candidats (x_m et $x_M = \beta_T(x_m)$) sont également l'unique solution de (1.20). En remplaçant T dans (1.20) par sa formule intégrale de (1.19) (sachant que $\bar{u} = \psi(\bar{x})$), nous faisons l'étude de la fonction $F : [a,b] \to \mathbb{R}$ définit par:

$$F(\alpha) := \int_{\alpha}^{\beta_T(\alpha)} \eta(x)(\psi(x) - \psi(\bar{x}))\,\mathrm{d}x.$$

Nous montrons dans la Proposition 3.2, que sous les hypothèses (1.9)-(1.11), cette fonction admet un unique zéro $\alpha = x_m$. Ce résultat entraîne immédiatement l'existence et l'unicité de $t \mapsto \hat{x}_T(t)$ solution de (1.6) avec $\hat{x}_T(0) = \hat{x}_T(T) = \bar{x}$, associée à un contrôle $\hat{u}_T$ défini par les deux temps de commutations $t_1, t_2 \in (0, T)$ comme suit:

$$\hat{u}_T(t) := \left| \begin{array}{rl} 1, & t \in [0, t_1), \\ -1, & t \in [t_1, t_2), \\ 1, & t \in [t_2, T]. \end{array} \right.$$

telles que $\hat{x}_T(t_1) = x_M$ et $\hat{x}_T(t_2) = x_m$ vérifiant (1.19)-(1.20).
Comme la paire (x_m, x_M), solution de (1.19)-(1.20), dépend de la valeur de T, on se pose la question sur la régularité et la monotonie des fonctions $T \mapsto x_m(T)$, $T \mapsto x_M(T)$. Dans le Lemme 3.4, nous montrons que ces fonctions sont de classe C^1 et sont respectivement décroissante et croissante. De plus, nous déduisons que:

$$\lim_{T \to +\infty} x_m(T) = a \quad \text{et} \quad \lim_{T \to +\infty} x_M(T) = b.$$

Après avoir caractérisé les différentes propriétés des trajectoires extrémales, l'étape suivante consiste à chercher les solutions optimales de (P_2). Pour répondre à cette question, nous commençons d'abord par établir un résultat fondamental du Chapitre 2, qui concerne l'étude du coût J_T associé aux trajectoires particulières (à deux commutations) dont nous avons parlé dans le paragraphe précédent. Afin de positionner le problème, prenons un exemple simple de fonctions $f, g, \ell : \mathbb{R} \to \mathbb{R}$ données par:

$$f(x) = -x^3 \ , \ g(x) = x \ , \ \ell(x) = x.$$

Dans ce cas $\psi(x) = x^2$ (et $\gamma = \psi$) est strictement convexe croissante dans $I = (0, 1)$ et les hypothèses (1.9)-(1.11)-(1.16) sont bien vérifiées dans cet intervalle. Pour $\bar{x} = 0.6$, nous traçons à l'aide du logiciel MATLAB, les solutions notées $x(\cdot)$ et $y(\cdot)$ associées à $\hat{u}_T$ et $\hat{u}_S$ respectivement avec $T = 4$ et $S = 6$ (voir la Fig. 1.2). La question qui se pose alors est de comparer les coûts $J_T(\hat{u}_T)$ et $J_S(\hat{u}_S)$. Nous montrons dans le Chapitre 2 le résultat suivant (voir Lemme 3.5).

Lemme 2.3. *Si les hypothèses* (1.9)-(1.16) *sont vérifiées, alors*

$$S > T > 0 \Rightarrow J_S(\hat{u}_S) < J_T(\hat{u}_T).$$

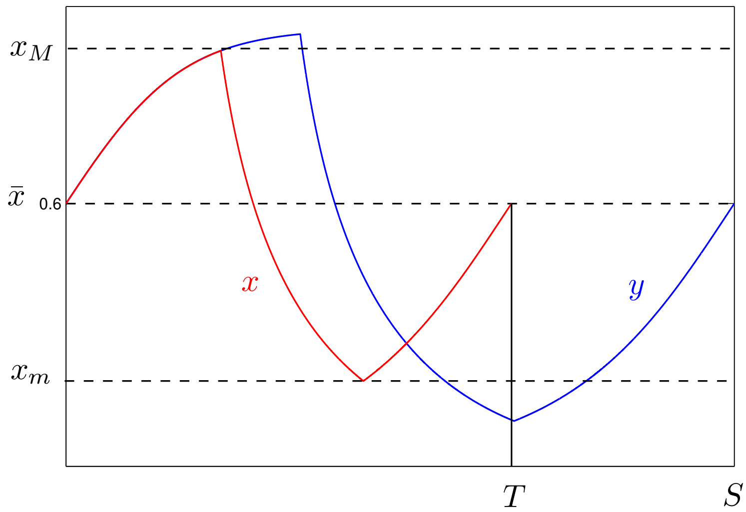


FIGURE 1.2 – *Solutions $x(\cdot)$ et $y(\cdot)$ associées à $\hat{u}_T$ et $\hat{u}_S$ avec $T = 4$, $S = 6$ et $\bar{x} = 0.6$.*

Pour démontrer ce résultat, on définit l'ensemble E qui contient les temps $t \in [0, S]$ pour lesquels la solution $y(\cdot)$ est en dehors de l'intervalle $[x_m, x_M]$ (voir la Fig. 1.2). Puisque les commandes $\hat{u}_T$ et $\hat{u}_S$ sont constantes par morceaux, on peut démontrer que:

$$\int_0^T \ell(x(t))\,\mathrm{d}t = \int_{[0,S]\setminus E} \ell(y(s))\,\mathrm{d}s.$$

De plus, la contrainte intégrale sur les deux commandes $\hat{u}_T$ et $\hat{u}_S$, qui peut s'écrire sous la forme (1.10), nous permet de déduire l'équation suivante:

$$\frac{1}{S-T}\int_E \gamma(\ell(y(s)))\,\mathrm{d}s = \bar{u}. \tag{1.21}$$

L'étape suivante consiste à définir une fonction intermédiaire $\hat{\gamma}(\cdot)$ à partir de la fonction $\gamma(\cdot)$ qui a la propriété d'être linéaire sur l'intervalle $[\ell(x_m), \ell(x_M)]$ inclus dans $\ell(I)$ et qui, en dehors de cet intervalle, coïncide avec γ (voir la Fig. 1.3). Cette fonction présente deux avantages:

- $\hat{\gamma}$ est linéaire vérifiant $\hat{\gamma} > \gamma$ sur $(\ell(x_m), \ell(x_M))$: cela implique

$$\hat{\gamma}\left(\frac{1}{T}\int_0^T \ell(x(t))\,\mathrm{d}t\right) = \frac{1}{T}\int_0^T \hat{\gamma}(\ell(x(t)))\,\mathrm{d}t > \frac{1}{T}\int_0^T \gamma(\ell(x(t)))\,\mathrm{d}t = \bar{u}.$$

↳ linéarité ↳ $\hat{\gamma} > \gamma$ ↳ (1.8)

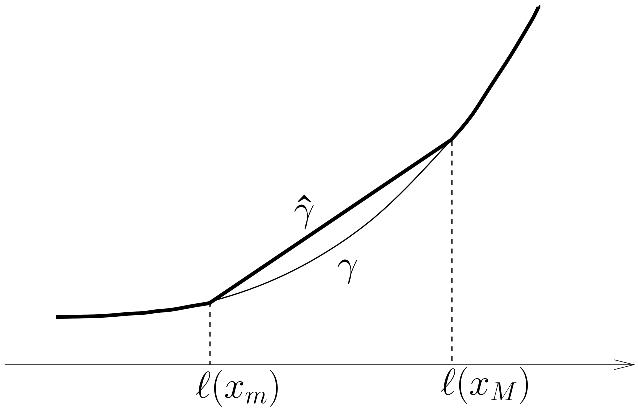


FIGURE 1.3 – *Graphe des fonctions γ et $\hat{\gamma}$.*

- $\hat{\gamma} = \gamma$ sur $\ell(I) \setminus [\ell(x_m), \ell(x_M)]$ et $\hat{\gamma}$ croissante sur $\ell(I)$: cette propriété implique

$$\frac{1}{S-T}\int_E \hat{\gamma}(\ell(y(s)))\,\mathrm{d}s = \bar{u} \implies \frac{1}{S-T}\int_E \ell(y(s))\,\mathrm{d}s \leq \hat{\gamma}^{-1}(\bar{u}).$$

Jensen + monotonie de $\hat{\gamma}$

En rassemblant toutes ces inégalités, nous obtenons:

$$\frac{1}{S}\int_0^S \ell(y(t))\,\mathrm{d}s < \frac{1}{T}\int_0^T \ell(x(t))\,\mathrm{d}t.$$

Nous sommes maintenant en mesure de déduire les solutions optimales du problème (P_2). Comme nous l'avons précisé précédemment, le coût associé à une extrémale à $2n$ commutations est égal au coût de sa restriction sur $[0, T/n]$. Pour un $T > 0$ fixé, la comparaison entre le coût associé à une extrémale ayant deux commutations (qui est le minimum de commutations possibles) sur $[0, T]$ et une autre ayant $2n$ commutations ($n > 1$) sur $[0, T]$ se fait à l'aide du lemme précédent sachant que $T/n < T$. Ces remarques nous aident à conclure le théorème fondamental de cette section qui est le Théorème 3.6 du Chapitre 2.

Théorème 2.4. *Sous les hypothèses* (1.9)-(1.16), *pour tout $T > 0$ fixé, une solution optimale du problème (P_2) est celle avec le nombre minimal de commutations, donnée par le contrôle $\hat{u}_T$ (ou une autre commande définie par translation de $\hat{u}_T$, en appliquant $-1, 1, -1$ au lieu de $1, -1, 1$).*

2.2.3 Relaxation des hypothèses

Les hypothèses considérées précédemment garantissent l'existence systématique d'un sur-rendement, quelque soit la valeur de $T > 0$. Or, il se peut que la fonction ψ ne vérifie pas l'hypothèse (1.11) sur tout l'intervalle I, ou bien, l'hypothèse (1.16) qui affirme l'existence d'un sur-rendement n'est pas valide sur tout l'intervalle I. Dans ces cas, un sur-rendement n'est pas nécessairement systématique. Dans le Chapitre 2, nous présentons également des conditions nécessaires pour l'optimalité de $\hat{u}_T$ (dans le Théorème 4.1) mais pour des valeurs restrictives de T.

Notons que l'hypothèse (1.11) est importante pour garantir l'existence et l'unicité de (x_m, x_M) vérifiant (1.19)-(1.20), indépendamment de la valeur de T. Remarquons que cette hypothèse implique clairement que $\psi'(\bar{x}) > 0$. Lorsque cette propriété (hypothèse (1.11)) est non-valide, nous demandons (dans la Proposition 4.1) à ce que la fonction ψ soit au moins croissante au voisinage de $\bar{x}$ c'est à dire $\psi'(\bar{x}) > 0$. Cela nous permet de déduire un sous intervalle de I contenant $\bar{x}$ et dans lequel la propriété $(\psi(x)-\psi(\bar{x}))(x-\bar{x})$ est vérifiée. Grâce aux propriétés des fonctions $T \mapsto x_m(T)$, $T \mapsto x_M(T)$, nous montrons que la paire (x_m, x_M) vérifiant (1.19)-(1.20) existe et est unique pour des périodes qui ne dépassent pas une valeur maximale T_{max}. Il est à noter que la valeur de T_{max} dépend des données du problème et peut être calculée numériquement.
Maintenant, après avoir affirmer l'existence et l'unicité des (x_m, x_M) vérifiant (1.19)-(1.20) pour des valeurs de $T \in (0, T_{max})$, on peut demander à ce que (1.16) soit valide dans (x_m, x_M) au lieu de I afin de conclure l'optimalité de $\hat{u}_T$. Toutefois, rien ne garantit que les trajectoires T-périodiques solutions de (1.6) et associées à $u \in [-1, 1]$ vérifiant (1.8) ne prennent pas des valeurs en dehors de (x_m, x_M). Pour contourner cette difficulté, nous proposons un résultat intermédiaire qui impose d'autres conditions sur la fonction ψ. Il s'agit de la Proposition 4.2. Finalement, l'optimalité de $\hat{u}_T$ pour les valeurs de $T \in (0, T_{max})$ est déduite grâce au Théorème 4.1. Pour des périodes au delà de T_{max}, on ne peut rien conclure sur l'optimalité des trajectoires bang-bang à deux commutations. Cette question est une perspective intéressante, qui prolongerait notre travail. Intuitivement, on s'attend à ce que le contrôle bang-bang ne reste pas optimal pour des périodes qui dépassent T_{max}. Tou-

tefois, pour $T = nT_{max}$, on peut garantir les mêmes performances obtenues avec T_{max} (en prenant des solutions T_{max}-périodiques).

2.3 Résumé du Chapitre 3

Comme nous l'avons expliqué au début de la Section 2.2, l'étude du problème (P_1) a été motivée par un exemple en bio-procédé concernant la dépollution des eaux usées. Le troisième chapitre de cette thèse sert à montrer comment se servir des résultats théoriques présentés dans le Chapitre 2 dans ce domaine applicatif.

Rappelons d'abord que le système (1.2) peut être réduit en un système uni-dimensionnel donné par (1.12) (en le considérant sur la variété invariante $\{s + x = s_{in}\}$) et qui peut être par la suite réécrit sous la forme du système différentiel (1.6) à l'aide d'un changement de variables adéquat. Il est à noter que toutes les cinétiques que nous avons citées dans la Section 1.2 sont des fonctions différentiables, positives qui s'annulent en $s = 0$. La fonction de type Contois peut être considérée comme fonction de s en remplaçant x par $s_{in} - s$. Pour une valeur constante du débit qu'on note $\bar{D} \in (D_{min}, D_{max})$, nous définissons l'équilibre (localement stable) de (1.12) associé à $\bar{D}$ par la formule:

$$\bar{s} := \inf\{s \ ; \ \nu(s) > \bar{D}\} < s_{in}.$$

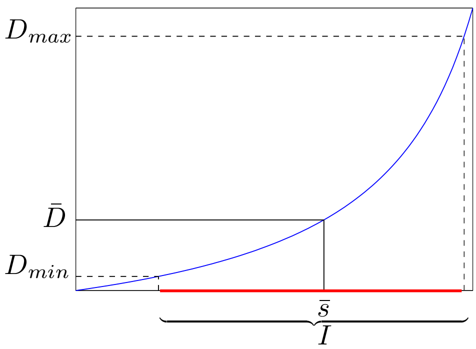
Dans ce chapitre, nous faisons l'étude des deux problématiques (Pb 1 et Pb 2) introduites au début de la Section 2.2. Tout d'abord, nous montrons dans le Lemme 3.1 que les deux problèmes admettent des solutions périodiques non-constantes vérifiant les contraintes $\frac{1}{T}\int_0^T D(t)\,\mathrm{d}t = \bar{D}$ ou $\frac{1}{T}\int_0^T s(t)\,\mathrm{d}t = \bar{s}$. Dans un second temps, en appliquant l'inégalité de Jensen, nous déduisons les conditions permettant de garantir un sur-rendement au sens de chaque problème. Ces conditions sont équivalentes aux hypothèses (1.16) et basées sur la convexité des fonctions μ.

Puisque dans le premier problème (Pb 1), nous imposons une contrainte intégrale sur la commande contrairement au deuxième problème (Pb 2), nous commençons d'abord à appliquer les résultats du Chapitre 2 sur ce premier problème afin de déduire les solutions optimales qui minimisent $\frac{1}{T}\int_0^T s(t)\,\mathrm{d}t$. Ceci est possible puisque la contrainte imposée dans le problème (Pb 1) doit être saturée, c'est à dire,

$$\frac{1}{T}\int_0^T D(t)\,\mathrm{d}t = \bar{D},$$

cela ayant été montré dans la Proposition 4.1.

Prenons comme exemple le modèle (1.12) avec μ de type Contois (en remplaçant x par $s_{in} - x$) qui est strictement convexe sur $\mathbb{R}_+$ lorsque $K > 1$ (voir la Fig. 1.4). Nous déduisons que, dans ce cas, pour toute valeur de $T > 0$, nous avons $\frac{1}{T}\int_0^T s(t)\,\mathrm{d}t < \bar{s}$. Les résultats de la Section 2.2.2 qui donnent les solutions optimales minimisant la moyenne en substrat s'appliquent (rappelons que la fonction $\gamma = \psi \circ \ell^{-1}$ est égale à ψ dans ce cas) (voir la Fig. 1.4). Les résultats du second chapitre abordés dans la Section



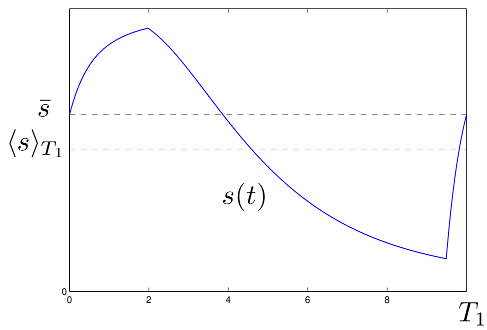


FIGURE 1.4 – *Graphe de la fonction $\mu(s, s_{in} - s)$ de type Contois (droite) et de la solution optimale (gauche) avec $\langle s\rangle_T := \frac{1}{T}\int_0^T s(t)\,\mathrm{d}t$.*

2.2.3 peuvent s'appliquer lorsque la cinétique μ dans (1.12) est de type Hill (c'est une fonction croissante et qui change de convexité sur $\mathbb{R}_+$). Sur la Fig. 1.5, nous présentons une situation où cette fonction n'est pas convexe sur tout l'intervalle I. Alors, pour certaines valeurs de T, nous déduisons l'optimalité des solutions bang-bang à deux commutations (et qui garantissent $\frac{1}{T}\int_0^T s(t)\,\mathrm{d}t < \bar{s}$). Si la valeur de T est trop grande ($T > \bar{T}$ sur la Fig. 1.6), le coût associée à cette commande pourrait croître et dépasser la valeur $\bar{s}$. Nous proposons dans le Chapitre 3, une loi de commande permettant d'avoir une concentration en substrat en moyenne toujours inférieur à $\bar{s}$, pour toute valeur de $T > \bar{T}$. Ce résultat est donné dans la Section 5 du

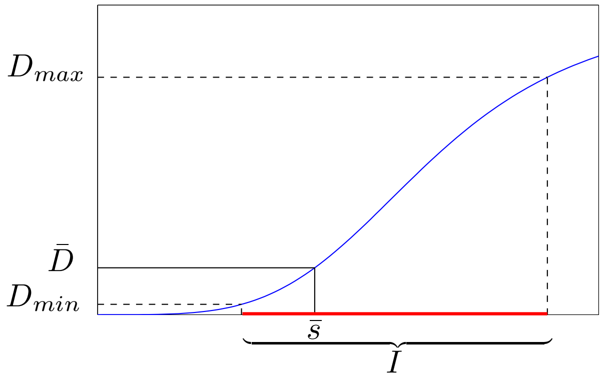


FIGURE 1.5 – *Graphe de la fonction μ de type Hill.*

Chapitre 3.
Dans ce qui précède, la valeur de $\bar{D}$ est supposée initialement fixée dans

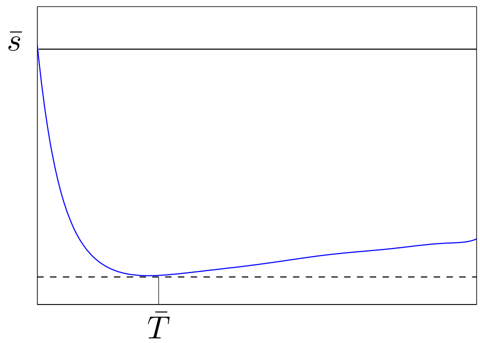


FIGURE 1.6 – *Coût associée à $\hat{u}_T$ pour μ de type Hill.*

(D_{min}, D_{max}). Rappelons que la solution optimale à $\bar{D}$ fixé est bang-bang et s'écrit:

$$\hat{D}(t) := \begin{cases} D_{max}, & 0 \leq t < t_1, \\ D_{min}, & t_1 \leq t < t_2, \\ D_{max}, & t_2 \leq t < T, \end{cases} \tag{1.22}$$

avec t_1 et t_2 sont les instants de commutations dans $(0, T)$. Si on pose $s_M = s(t_1)$ et $s_m = s(t_2)$ alors $(s_m, s_M) \in I^2$ vérifie (1.19)-(1.20) (en tenant compte des changements de variables cités ci-dessus).

Il est intéressant d'un point de vue mathématique et applicatif d'étudier la variation du coût optimal par rapport à $\bar{D}$. Pour cette raison, nous introduisons dans le Chapitre 3 la fonction "valeur" $\bar{D} \mapsto V_T(\bar{D})$ dans (D_{min}, D_{max})

comme suit:

$$V_T(\bar{D}) := \min_{D(\cdot)} \left\{ \frac{1}{T}\int_0^T s(t)\,\mathrm{d}t\,;\; s(0)=s(T),\; \frac{1}{T}\int_0^T D(t)\,\mathrm{d}t = \bar{D} \right\}.$$

Nous montrons dans le Théorème 4.1 que la fonction V_T est croissante. En outre, à l'aide des résultats de la théorie du contrôle optimal, nous montrons que lorsque les hypothèses (1.16) sont vérifiées pour ce modèle, la fonction V_T est continue dans (D_{min}, D_{max}), pour toute valeur de $T > 0$. Cette propriété résulte du fait que l'application $\hat{D}(\cdot)$ est continue dans L^1 par rapport à $\bar{D}$. De plus, l'application $D(\cdot) \mapsto s(\cdot, D)$ est continue de L^1 dans C^0 comme les conditions du Théorème 4.2 cité dans [BP07] sont valides dans notre cas. Par conséquent, nous déduisons que $\bar{D} \mapsto s(\cdot, \hat{D}_T)$ est continue et ainsi la fonction $V_T(\cdot)$ l'est.

La recherche des solutions optimales minimisant $\frac{1}{T}\int_0^T D(t)\,\mathrm{d}t$ (pour le problème Pb 2) est liée au premier. Nous définissons ainsi la fonction $\bar{s} \mapsto W_T(\bar{s})$ dans I comme:

$$W_T(\bar{s}) := \max_{D(\cdot)} \left\{ \frac{1}{T}\int_0^T D(t)\,\mathrm{d}t\,;\; s(0)=s(T),\; \frac{1}{T}\int_0^T s(t)\,\mathrm{d}t = \bar{s} \right\}.$$

Nous montrons la *dualité* entre les deux problèmes du fait de la monotonie de la fonction $V_T(\cdot)$ et qui se traduit dans le résultat suivant (voir la Proposition 4.3 du Chapitre 3).

Proposition 2.3. *Pour tout $\bar{s} \in I$ et $T > 0$ tels que la solution du Problème 1 existe, alors $W_T(\bar{s}) = V_T^{-1}(\bar{s})$.*

Ce résultat permet de conclure que la contrainte intégrale du deuxième problème (Pb 2) doit être saturée et d'en déduire également la solution optimale maximisant $\frac{1}{T}\int_0^T s(t)\,\mathrm{d}t$ qui est également bang-bang à deux commutations. Cette synthèse résulte par construction à partir de la solution du premier problème. Soit α (qui joue le rôle de $\bar{s}$) choisi dans l'image de l'intervalle (D_{min}, D_{max}) par V_T et qui représente la moyenne de la solution optimale associée à la commande $\hat{D}(\cdot)$ ayant β comme moyenne (voir la Fig. 1.7). Cette dernière (la constante β) est égale exactement à $W_T(\alpha)$.

Nous nous intéressons ensuite à simuler numériquement les fonctions

V_T et W_T, en utilisant le résultat précédant, pour le modèle (1.12) avec une cinétique de type Contois. Il s'agit de calculer pour chaque valeur de $\bar{D} \in (D_{min}, D_{max})$, la solution de (1.12) vérifiant $s(0) = s(T) = \bar{s}$ et la contrainte $\frac{1}{T}\int_0^T D(t)\,\mathrm{d}t = \bar{D}$. Comme la solution optimale est constante par morceaux, la contrainte sur le débit permet d'écrire une relation entre les deux temps de commutations t_1 et t_2:

$$t_2 = t_2(t_1) := t_1 + T\frac{D_{max} - \bar{D}}{D_{max} - D_{min}}.$$

Il suffit alors de rechercher l'instant $t_1 \in (0, T)$ pour lequel $s(T) - \bar{s} = 0$. Puisque le coût optimal est inférieur à $\bar{s} = \mu^{-1}(\bar{D})$ pour chaque $\bar{D}$ fixé, le graphe de la fonction valeur V_T se situe en dessous de celui de μ^{-1} (voir la Fig. 1.7).

La détermination des instants de commutations est beaucoup plus compliquée pour le problème *dual* (Pb 2) car il admet une contrainte intégrale sur l'état, d'où l'intérêt de passer par la formulation *duale* donnée dans la Proposition 2.3. D'autres simulations similaires dans le cas où μ est de type

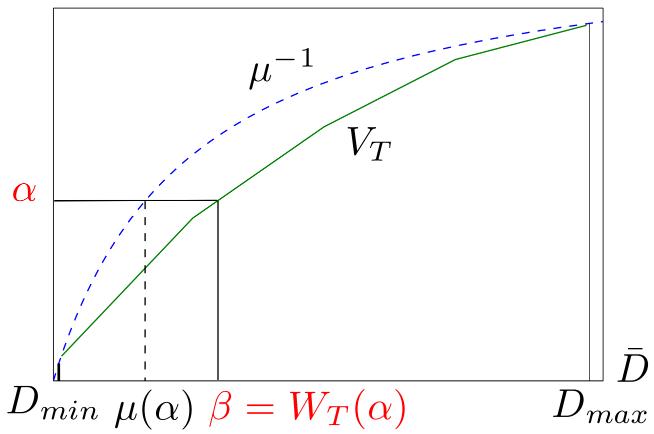


FIGURE 1.7 – *Représentation de V_T et W_T avec T fixé (pour système (1.12) avec fonction de type Contois).*

Hill sont présentées dans la Section 5 du Chapitre 3. Nous calculons également le pourcentage de gain en appliquant les commandes périodiques versus les commandes constantes pour différentes valeurs de T et de $\bar{D}$.

2.4 Résumé du Chapitre 4

Le but du Chapitre 4 est de montrer comment exploiter les résultats d'existence de sur-rendement dont nous avons parlé dans la Section 2.3, pour caractériser la loi de croissance que suivent les espèces dans un bioréacteur (supposée inconnue). Parmi les méthodes classiques permettant d'estimer la cinétique μ est de réaliser une série d'expériences avec différentes valeurs constantes de D, afin de reconstituer le graphe de cette fonction. Cette méthode présente plusieurs inconvénients puisqu'il pourrait exister plusieurs équilibres dépendamment de la valeur de D et de la monotonie de μ. Une autre difficulté se pose lorsque la fonction de croissance est densité dépendante. Si c'est le cas, il est impératif de mesurer à la fois la concentration en biomasse et en substrat, ce qui peut s'avérer très difficile voire impossible parce que trop coûteux.

Nous présentons dans ce chapitre une méthode qualitative et robuste permettant de distinguer entre deux types de cinétiques: une loi de Monod ou de Contois avec $K < 1$ et une loi de Contois avec $K > 1$. Nous proposons alors deux terminologies: « effet densité faible » lorsqu'il s'agit d'une loi de Monod ou de Contois avec $K < 1$ et « effet densité fort » lorsqu'il s'agit d'une loi de Contois avec $K > 1$. En se basant sur les résultats du Chapitre 2, nous proposons un algorithme qui permet de conclure à quelle catégorie appartient la courbe de croissance inconnue et ceci de façon robuste. Supposons l'existence d'un seul substrat et d'une seule espèce dans le milieu réactionnel suivant le modèle (1.12), la procédure proposée dans le Chapitre 4 se résume en les étapes suivantes (voir la Fig. 1.8):

1. Conduire, pour un choix de D adéquat, le système (1.12) vers un équilibre positif s^{eq}.
2. Choisir deux valeurs constantes D_{min} et D_{max} tels que $D_{min} < D < D_{max}$ et appliquer une commande bang-bang (qui commute deux fois entre D_{min} et D_{max}) afin de ramener la solution s vers s^{eq} après une période choisie $T > 0$.
3. Calculer la moyenne de la commande appliquée (qu'on note $\bar{D}$) dans l'étape précédente ainsi que la moyenne de s (qu'on note $\hat{s}$).

4. Appliquer $\bar{D}$ et attendre que le système se stabilise (vers un certain $s^\star$). La comparaison entre $\hat{s}$ et $s^\star$ permet de conclure. En effet, si on trouve que $\hat{s} < s^\star$ alors on valide l'effet densité fort, par contre si $\hat{s} > s^\star$, on valide l'effet densité faible.

L'algorithme proposé présente un autre intérêt principal qui repose sur sa robustesse vis-à-vis des perturbations. En effet, il est bien connu que les cultures continues sont souvent sujettes à des changements environnementaux incontrôlés (changement de pH, de température, etc) qui ont un impact en particulier sur le taux de croissance maximal μ_{max}.
Nous prenons en compte dans la Section 5 du Chapitre 4 deux types de perturbations: changement soudain de s_{in} et de μ_{max} et calculons $\Delta s := \hat{s} - s^\star$. Nous observons et vérifions numériquement que Δs reste de signe constant et ne s'affecte pas par ces différentes variations d'où la robustesse du test proposé.

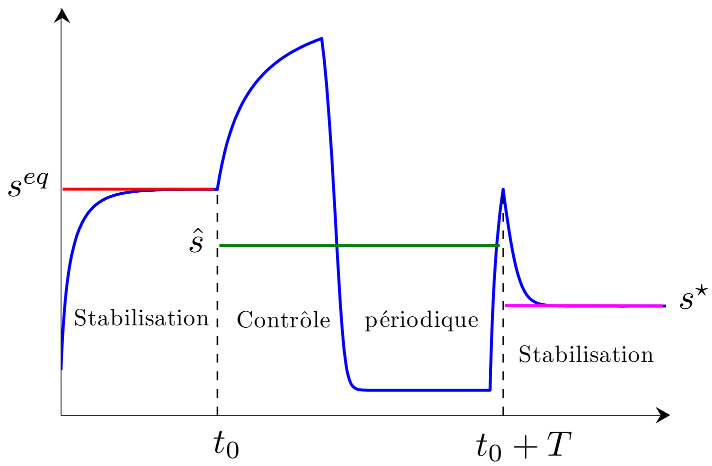


FIGURE 1.8 – *Schématisation de la procédure.*

3 Résilience faible dans le chémostat à l'aide du contrôle périodique

Le terme « résilience » qui signifie « rebondir » ou « revenir à l'état initial

après une perturbation » est employé généralement dans le domaine de la physique, de la psychologie et de l'écologie. En 1973, Holling dans [Hol73] définit l'*ecological resilience* comme la capacité d'un écosystème à s'adapter et se reconstruire après avoir subi des perturbations. D'une manière générale, la notion de perturbation dans un environnement écologique peut être liée à la prédation, à l'invasion de nouvelles espèces, à des changements climatiques ou bien à des changements des paramètres environnementaux (pH, température, etc).
Afin de maintenir la résilience d'un système dynamique, il est possible d'appliquer des stratégies de contrôle (ou politiques d'actions) qui ramènent le système dans un ensemble d'états vérifiant des propriétés désirées, comme expliqué dans [DGG11]. Ceci nous amène à parler de « la théorie de viabilité » (voir [Aub09]). En effet, l'originalité de la théorie de viabilité est de s'intéresser à des systèmes contrôlés du type:

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)), \\ u(t) \in U(x(t)), \end{cases} \tag{1.23}$$

prenant en compte le fait que les différents contrôles u qui peuvent intervenir au temps t appartiennent à un ensemble $U(x(t))$ (dans $\mathbb{R}^m$) qui dépend de l'état du système au temps t. De tout point initial, il peut exister plusieurs évolutions possibles de la trajectoire solution $x(\cdot)$ suivant les différentes valeurs des contrôles choisies au cours du temps. Un ensemble de contraintes $K \subset \mathbb{R}^n$ est dit viable pour le système contrôlé (1.23), si pour toute condition initiale x_0 dans K, il existe une solution de (1.23) partant de x_0 et qui reste dans K pour tout temps ultérieur. Une solution du système (1.23) qui vérifie cette propriété s'appelle alors trajectoire viable. Les méthodes et outils de la théorie de la viabilité permettent de rechercher les conditions (décisions, états) pour lesquelles l'ensemble de contraintes K, qui regroupent les propriétés désirées, est viable. Les applications sont nombreuses en écologie, économie ou robotique, lorsqu'un système dépérit ou se détériore en quittant une certaine zone de son espace d'état. Il est à noter que la théorie de viabilité classique ne renseigne pas du retour possible vers K lorsque la condition initiale est en dehors de K.
Revenons maintenant aux travaux de [Mar05] et [DGG11]. Des définitions mathématiques de résilience écologique ont été proposées, compatibles avec

la définition conceptuelle de Holling. Ces formulations mathématiques sont basées essentiellement sur les concepts des attracteurs de systèmes dynamiques et de la théorie de viabilité . Dans [DGG11], les auteurs expliquent que la définition basée sur l'approche de viabilité exprime mieux la signification originelle de résilience (introduite dans [Hol73]) et donne une interprétation mathématique précise. De plus, il est possible d'utiliser les algorithmes de calcul de noyau de viabilité [9] et du bassin de capture [10] pour donner des valeurs approximatives de résilience. Toutefois, ces méthodes numériques présentent certaines limitations ; notamment, elles souffrent de la malédiction de la dimensionnalité et leur application est réservée à des problèmes en petite dimension (dans l'espace d'état et des contrôles) [Cha07].

En micro-biologie, les populations de micro-organismes subissent différentes perturbations qui peuvent conduire à des changements considérables de leurs développements. Nous nous intéressons dans la deuxième partie de cette thèse à l'influence de l'invasion de nouvelles espèces de micro-organismes dans un chémostat. Cette invasion biologique démarre généralement par l'introduction d'une population en petits effectifs dans le milieu. Si les conditions sont favorables pour sa croissance, elle se propage, s'installe et envahit ce milieu aux détriments des espèces résidentes. Le problème majeur que causent ce type de perturbations est soit l'extinction des espèces natives [11] à cause de la compétition entre espèces (nous rappelons plus loin le principe de l'exclusion compétitive), ou les maintenir à de faibles concentrations pour leur survie. Cette partie de thèse vise le cas où une population de micro-organismes est présente à l'équilibre dans le milieu de culture alimenté à débit constant, avant l'invasion d'une nouvelle espèce. Après l'apparition d'une nouvelle population de micro-organismes (considérée comme perturbation), nous verrons que sous certaines conditions sur les taux de croissance des deux espèces, la propriété de résilience (au sens qualitatif de [Hol73]) est perdue si la variable de commande (le

9. Le noyau de viabilité d'un ensemble fermé $K \subset \mathbb{R}^N$ noté $Viab(K)$ est le plus grand sous ensemble viable de K pour le système contrôlé (1.23) (voir [Aub09]).

10. Soit C une cible dans K. Le bassin de capture du système (1.23) noté $Capt(C, K)$, représente l'ensemble des états initiaux x_0 pour lesquels il existe une commande mesurable u tel que $x(\cdot, x_0, u)$ atteint la cible C en un temps fini $T > 0$ (voir [Aub09]).

11. Ce sont les espèces présentes dans le milieu de culture avant l'invasion.

débit dans notre cas) reste constante. En lien avec le concept de viabilité, nous montrons qu'il n'existe pas de trajectoires viables, contenues dans un ensemble de contraintes bien défini, en présence d'une nouvelle espèce.

L'objectif principal auquel nous nous intéressons est de définir une commande (le débit D) non-constante pour le modèle de chémostat, de manière à restaurer une « faible résilience ».

Nous allons proposer une définition mathématique de « faible résilience » pour notre cas d'étude. Cela consiste à déterminer une loi de commande (le débit D) qui permet aux trajectoires du système de revenir une infinité de fois à l'ensemble des contraintes et même de façon asymptotiquement périodique (sans jamais pouvoir y rester). C'est pour cela qu'on accorde le qualificatif « faible » au mot « résilience ». Nous précisons que le but ici n'est pas de choisir une solution « optimale » en fonction d'un certain critère, mais de proposer une loi de commande dite « faiblement résiliente ».

3.1 Résumé du Chapitres 5 et 6

Avant de présenter les étapes permettant d'atteindre l'objectif fixé précédemment, nous commençons à donner un bref aperçu sur le principe d'exclusion compétitive pour deux espèces. Cela nous sera nécessaire pour construire une loi de commande dite « faiblement résiliente ».

Prenons le modèle simple de chémostat à deux espèces:

$$\left\{ \begin{array}{lcl} \dot{s}(t) & = & D(s_{in} - s) - \mu_1(s)x_1 - \mu_2(s)x_2, \\ \dot{x}_1(t) & = & (\mu_1(s) - D)x_1, \\ \dot{x}_2(t) & = & (\mu_2(s) - D)x_2, \end{array} \right. \tag{1.24}$$

la fonction μ_i représente le taux de croissance de l'espèce i supposée par exemple de type Monod

$$\mu_i(s) := \frac{m_i s}{K_i + s}\ ,\ i = 1, 2$$

Pour chaque i, cette fonction est définie pour s positif, continue, nulle en 0, croissante et bornée.

Rappelons que pour des conditions initiales positives, les solutions du système (1.24) restent positives au cours du temps. En plus, la théorie mathématique du chémostat affirme que l'ensemble $\{(s, x_1, x_2) \in \mathbb{R}^3_+ \ , \ s + x_1 + x_2 = s_{in}\}$ est invariant et attractif dés que D est non nul. On définit la notion suivante (voir [HLRS17]).

Définition 3.1. *Pour une valeur de D fixée, on appelle seuil de croissance $\lambda_i(D)$ pour l'espèce i la concentration s vérifiant $\mu_i(s) = D$. S'il n'existe pas de solution à cette équation, on pose $\lambda_i(D) = +\infty$.*

Puisque nous prenons ici des cinétiques de type Monod alors l'équation $\mu_i(s) = D$ admet une unique solution lorsque $D < m_i$ qui est donnée par:

$$\lambda_i(D) := \frac{DK_i}{m_i - D}.$$

Il est facile de vérifier que les graphes des deux fonctions de croissance peuvent s'intersecter en dehors de 0, en un point $\bar{s} \in (0, +\infty)$. Dans la suite, nous expliquons brièvement le principe de compétition entre espèces, en fonction de la valeur de D. Ces résultats théoriques sont expliqués en détail dans [HLRS17].

- Cas 1: $0 < D < \max_{i=1,2} \mu_i(s_{in})$: Il faut d'abord mentionner que dans ce cas, il existe au moins un seuil de croissance λ_i qui est inférieur à s_{in}. Le théorème de l'exclusion compétitive montre que dans ce cas, la seule espèce qui survit est celle qui a le plus faible seuil de croissance et l'autre disparaît. En d'autres termes, si $\lambda_1(D) < \lambda_2(D)$, l'équilibre $E_1 := (\lambda_1(D), s_{in} - \lambda_1(D), 0)$ est globalement asymptotiquement stable. Sur la Fig.1.9 et Fig.1.10, nous illustrons cette propriété pour différentes valeurs de paramètres.
  Dans le cas où $\lambda_2(D) < \lambda_1(D)$, c'est l'équilibre $E_2 := (\lambda_2(D), 0, s_{in} - \lambda_2(D))$ qui est globalement asymptotiquement stable. Nous présentons également sur la Fig.1.11 et Fig.1.12 d'autres simulations qui illustrent ce cas. La stabilité exponentielle locale se démontre facilement en calculant les valeurs propres de la jacobienne et la stabilité globale se démontre en introduisant une fonction de Liapunov.
  Une autre situation très particulière concerne le cas où les graphes des deux fonctions μ_i se croisent en $\bar{s} \in (0, s_{in})$ avec un taux de dilu-

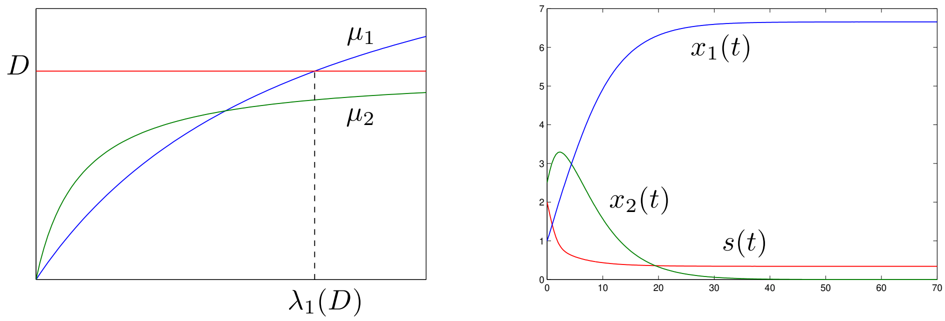


FIGURE 1.9 – *Graphes des fonctions* μ_i *(gauche) et solution du système* (1.24) *(droite) avec* $D = 1$, $m_1 = 2$, $m2 = 1$, $K_1 = 5$, $K_2 = 0.8$, $s_{in} = 7$.

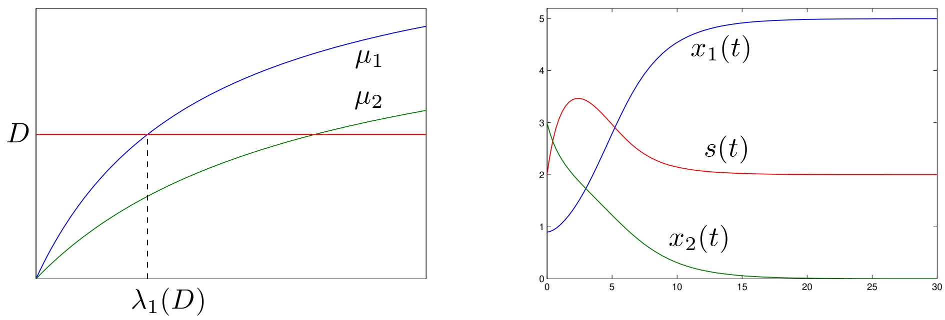


FIGURE 1.10 – *Graphes des fonctions* μ_i *(gauche) et solution du système* (1.24) *(droite) avec* $D = 0.8$, $m_1 = 2$, $m2 = 1.6$, $K_1 = 3$, $K_2 = 5$, $s_{in} = 7$.

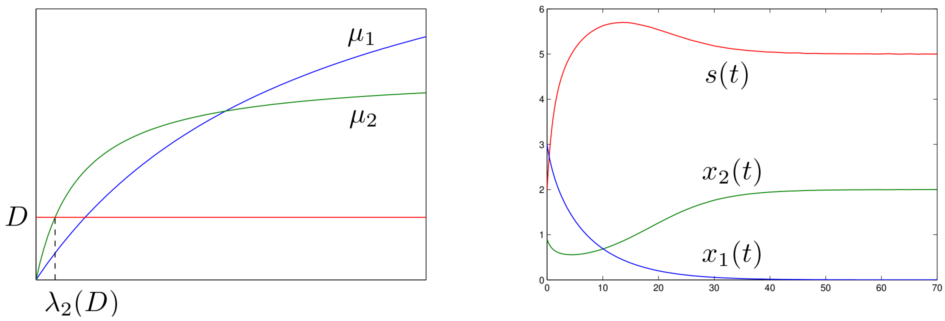


FIGURE 1.11 – *Graphes des fonctions* μ_i *(gauche) et solution du système* (1.24) *(droite) avec* $D = 0.3$, $m_1 = 2$, $m2 = 1$, $K_1 = 5$, $K_2 = 0.8$, $s_{in} = 7$.

tion D choisi exactement en $\mu_1(\bar{s})$ (qui est égal à $\mu_2(\bar{s})$). Ce cas désigne également l'égalité des seuils de croissance. Dans [HHW77], les auteurs ont établi que dans ce cas, la coexistence des deux populations est possible c'est à dire que les solutions du système (1.24) convergent vers un équilibre de la forme $(\bar{s}, x_1^e, x_2^e)$ avec $x_1^e + x_2^e = s_{in} - \bar{s}$ (dit équilibre de coexistence). Ceci a été confirmé dans [Kee83] par des techniques de bifurcation. Rechercher les situations qui permettent

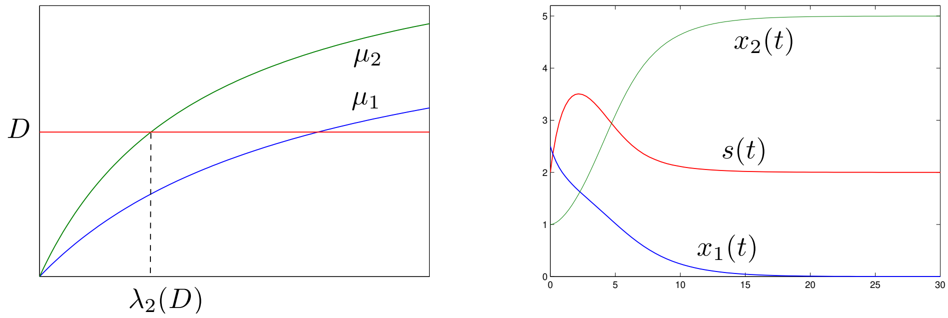


FIGURE 1.12 – *Graphes des fonctions* μ_i *(gauche) et solution du système* (1.24) *(droite) avec* $D = 0.8$, $m_1 = 1.6$, $m2 = 2$, $K_1 = 5$, $K_2 = 3$, $s_{in} = 7$.

aux espèces en compétition de coexister est un des objectifs qui intéressent de nombreux chercheurs en écologie. Toutefois, le cas d'égalité des seuils de croissance n'est jamais satisfait en pratique, c'est un cas non générique. En effet, ceci signifie qu'il faut maintenir le taux de dilution D à une valeur précise ce qui n'est pas possible en pratique à cause des perturbations. Ainsi, mise à part cette valeur particulière de D, le modèle (1.24) n'admet pas d'équilibre avec plus d'une espèce.

- Cas 2: $D \geq \max\limits_{i=1,2} \mu_i(s_{in})$: Une analyse du modèle (1.24) permet de conclure que $x_1(t)$ et $x_2(t)$ convergent vers la solution nulle lorsque t tend vers l'infini et $s(t)$ converge vers s_{in}. L'équilibre $E_0 := (s_{in}, 0, 0)$ est le seul équilibre globalement asymptotiquement stable dans l'orthant positif et s'appelle « équilibre de lessivage ». Sur la Fig. 1.13, nous avons simulé quelques trajectoires.

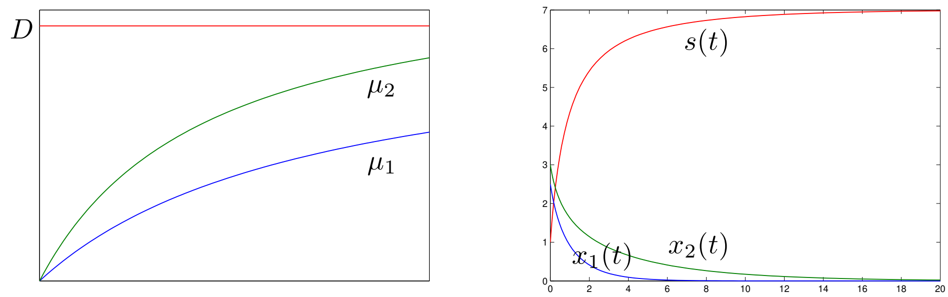


FIGURE 1.13 – *Graphes des fonctions* μ_i *(gauche) et solution du système* (1.24) *(droite) avec* $D = 1.6$, $m_1 = 1.6$, $m2 = 2$, $K_1 = 5$, $K_2 = 3$, $s_{in} = 7$.

Dans cette étude, nous considérons un problème d'invasion. Nous supposons que le milieu de culture dans le chémostat contient une espèce (de concentration x_1) qui consomme un substrat. Nous supposons aussi que le dispositif expérimental est alimenté avec un débit constant $D > 0$ et que le système est stabilisé autour d'un équilibre (s^{eq}, x_1^{eq}) dans $\mathbb{R}_+^\star$. A ce moment, nous considérons qu'une nouvelle espèce à faible concentration (x_2) apparaît dans le milieu réactionnel supposé favorable pour qu'elle s'installe et envahisse ce milieu. Nous supposons aussi que les cinétiques μ_1 et μ_2 sont croissantes et s'intersectent en un point $\bar{s} \in (0, s_{in})$ (comme sur la Fig. 1.11). La concentration initiale $x_2(0)$ étant supposée très faible, nous supposons que la quantité $s(0) + x_1(0) + x_2(0)$ est quasiment égale à s_{in}. Si la valeur de D est choisie dans $(0, \mu_1(\bar{s}))$ alors d'après la brève discussion précédente sur la compétition des espèces, les solutions du système (1.24) convergent vers l'équilibre E_2. On dit alors que le système est « non-résilient » (puisque x_1 tend vers 0) lorsque la concentration de l'espèce native à l'équilibre avant l'invasion, est au dessous de $s_{in} - \bar{s}$ (qui est équivalent à dire que D est dans $(0, \mu_1(\bar{s}))$. Nous expliquons dans la suite comment montrer ce résultat mathématiquement. Tout d'abord, choisissons un seuil x_1^r dans $(s_{in} - \bar{s}, s_{in})$. Ensuite, nous introduisons l'ensemble $\mathcal{K}(x_1^r)$ comme:

$$\mathcal{K}(x_1^r) := \{x \in \mathbb{R}_+^2 \ ; \ x_1 + x_2 \leq s_{in} \ ; \ x_1 \geq x_1^r \ \text{ and } \ x_2 > 0\},$$

et fixons les paramètres D_m et D_M tels que:

$$0 < D_m < \mu_1(s_{in} - x_1^r) < \mu_1(\bar{s}) < \mu_1(s_{in}) < D_M. \tag{1.25}$$

En considérant la dynamique du système réduit:

$$\begin{cases} \dot{x}_1(t) = (\mu_1(s_{in} - x_1 - x_2) - D)x_1 \\ \dot{x}_2(t) = (\mu_2(s_{in} - x_1 - x_2) - D)x_2 \end{cases} \tag{1.26}$$

nous montrons le résultat suivant en utilisant la théorie de viabilité (Lemme 2.1 du Chapitre 5).

Lemma 3.1. *Le noyau de viabilité de l'ensemble $\mathcal{K}(x_1^r)$ pour la dynamique* (1.26) *est tel que:*

$$Viab(cl\,\mathcal{K}(x_1^r)) = [x_1^r, s_{in}] \times \{0\}.$$

La preuve de ce lemme se fait en deux parties. En premier lieu, nous montrons que lorsque la condition initiale $x_{2,0}$ est nulle alors la trajectoire $x(\cdot)$ solution de (1.26) reste dans $\mathcal{K}(x_1^r)$. Ensuite, en prenant une condition initiale $x_{2,0}$ non nulle, nous supposons par absurde qu'une solution $x(\cdot)$ de (1.26) reste dans $\mathcal{K}(x_1^r)$. Nous prouvons dans ce cas que la solution $x_1(t)$ décroît vers zéro ce qui est une contradiction. Cela implique en particulier qu'en présence de l'espèce 2, l'ensemble $\mathcal{K}(x_1^r)$ n'est pas viable avec un débit $D \in [D_m, D_M]$ constant.

Notons que dans ce travail, on cherche à préserver l'espèce 1 à un niveau suffisant pour son intérêt industriel (contrairement à l'espèce 2). Nous cherchons ainsi à examiner la possibilité de ramener x_1 au dessus de x_1^r avec un débit D qui varie au cours du temps et qui alterne entre les deux valeurs D_m et D_M vérifiant (1.25). Aucune de ces valeurs n'est favorable pour l'espèce 1, la constante $D = D_M$ est défavorable pour les deux espèces et $D = D_m$ est favorable pour l'espèce envahissante, mais nous montrons que le résultat final peut être avantageux pour l'espèce d'intérêt.

Nous définissons mathématiquement la notion de « faible résilience » comme suit:

Définition 3.2. *Une fonction $D(\cdot)$ appartenant à $[D_m, D_M]$ (vérifiant* (1.25)*) est dite faiblement résiliente si les solutions de* (1.26) *associées à $D(\cdot)$ vérifient*

$$\text{mes}\,\{t \geq 0\ ;\ x(t) \in \mathcal{K}(x_1^r)\} = +\infty.$$

La construction de telle fonction $D(\cdot)$ est basée principalement sur les propriétés asymptotiques des solutions du (1.26) lorsque $D = D_M$ et $D = D_m$ (vérifiant (1.25)). En effet,

- quand $D = D_M$, l'équilibre de lessivage $\tilde{E}_0 = (0, 0)$ est globalement asymptotiquement stable,
- pour $D = D_m$, l'équilibre $\tilde{E}_2 = (0, s_{in} - \lambda_2(D_m))$ est un nœud stable alors que $\tilde{E}_1 = (s_{in} - \lambda_1(D_m), 0)$ est un point selle. Dans ce cas, nous déterminons les variétés invariantes (stable et instable) de la dynamique (1.26).

Nous illustrons les portraits de phase de (1.26) (avec $D = D_M$ et $D = D_m$)

sur la Fig. 1.14. Remarquons que pour une valeur de $x_2(0)$ suffisamment

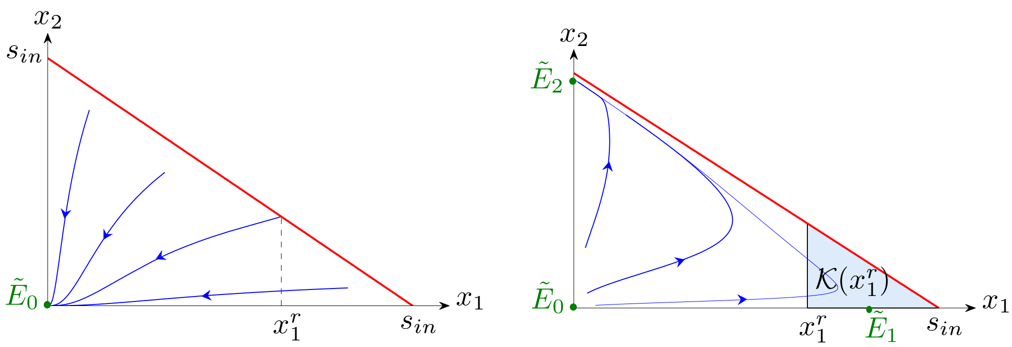


FIGURE 1.14 – *Portraits de phase de* (1.26) *avec* $D = D_M$ *(gauche) et* $D = D_m$ *(droite).*

petite, les trajectoires avec $D = D_m$ pénètrent l'ensemble $\mathcal{K}(x_1^r)$. Alors, le premier résultat principal de ce chapitre, développé dans la Proposition 3.1, consiste à proposer une fonction $D(\cdot)$ qui ramène les trajectoires suffisamment proche de l'origine avec $D = D_M$, puis, prend la valeur D_m pour les ramener à $\mathcal{K}(x_1^r)$. Plus précisément, on a le résultat suivant.

Proposition 3.3. *Soit* $\varepsilon > 0$ *suffisamment petit et soit l'ensemble* $\mathcal{E} := (0, \bar{x}_1] \times (0, \varepsilon]$. *La fonction* $D(\cdot)$ *définie comme suit:*

- $D(t) = D_M$ *pour* $t \in [T_i, T_{i+1})$ *si* $x(T_i) \notin \mathcal{E}$ *et* T_{i+1} *est le premier instant pour atteindre* $\mathcal{E}$*,*
- $D(t) = D_m$ *pour* $t \in [T_i, T_{i+1})$ *si* $x(T_i) \in \mathcal{E}$ *et* T_{i+1} *est le premier instant pour sortir de* $\mathcal{K}(x_1^r)$*,*

est faiblement résiliente où $x(\cdot)$ *est solution de* (1.26) *associée* $D(\cdot)$ *avec* $x(0) \in \mathcal{K}(x_1^r)$*. Les instants* T_i *forment une suite croissante qui tend vers* $+\infty$*.*

Il est à noter que c'est la tangence des courbes solutions avec l'axe $x_2 = 0$ lorsque $D = D_M$ en des temps grands qui nous permet de montrer une telle propriété. Cette tangence est expliquée par le fait que la croissance μ_1 est supérieure à celle de μ_2 pour des valeurs de s grandes, c'est à dire lorsque le système est poche du lessivage (équilibre $\tilde{E}_0$).

La deuxième partie du Chapitre 5 est consacrée à montrer d'abord l'existence d'une solution périodique (qu'on note $x^\star(\cdot)$) associée à $D(\cdot)$ et ensuite à prouver que toute autre solution converge asymptotiquement vers $x^\star(\cdot)$.

Avant de détailler notre démarche, notons que dans ce travail, les instants T_i ne sont pas fixés à l'avance et dépendent nécessairement de la condition initiale. Donc, la période associée à $x^\star(\cdot)$ n'est pas fixée. Par conséquent, l'usage de l'application de Poincaré considérée comme outil d'étude de l'existence et de la stabilité de solutions périodiques (comme fait dans [SW95]) n'est pas possible dans notre cas.
Considérons une condition initiale particulière $(x_1^r, x_{2,0})$ et définissons l'opérateur $\mathcal{O}$ comme

$$\begin{array}{rccc} \mathcal{O}: & (0, s_{in} - x_1^r] & \to & (0, s_{in} - x_1^r] \\ & x_{2,0} & \mapsto & x_2(T_2) \end{array}$$

où $x(\cdot) = (x_1(\cdot), x_2(\cdot))$ est solution de (1.26) associée à $D(\cdot)$ et $x_1(T_2) = x_1^r$. L'instant T_2 est défini dans la Proposition 3.3 (voir la Fig. 1.15). Nous mon-

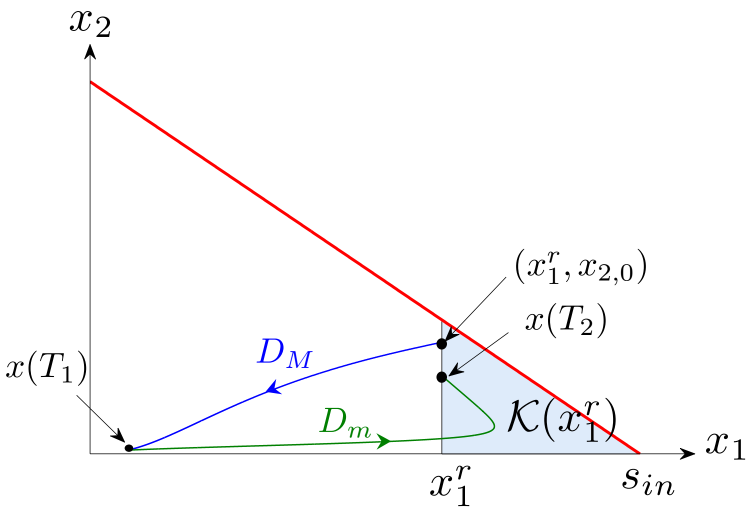


FIGURE 1.15 – *Présentation d'une orbite de* (1.26) *associée à* $D(\cdot)$ *défini dans la Proposition 3.3 sur* $[0, T_2]$.

trons dans la Section 4 du Chapitre 5 le résultat suivant (voir les Propositions 4.1-4.2).

Proposition 3.4. *L'opérateur $\mathcal{O}$ est Lipschitz continu et décroissant.*

Cet opérateur peut être écrit comme fonction des instants T_1 et T_2 qui dépendent de la condition initiale $x_{2,0}$. Nous montrons dans un premier temps que $T_1(\cdot)$ et $T_2(\cdot)$ sont Lipshitz continues par rapport à $x_{2,0}$. Cela se fait à l'aide d'un résultat en théorie du contrôle sur les fonctions appelées *first entry time functions* [12] qu'on note $R_1(\cdot)$ et $R_2(\cdot)$. Ces deux fonctions vérifient la *condition de Petrov* (nous rappelons ce résultat dans l'Appendice 6) et donc elles sont Lipshitz continues [CS04]. Par conséquent, nous déduisons la continuité de $T_1(x_{2,0})$ et $T_2(x_{2,0})$ et ainsi la continuité de $\mathcal{O}$ par construction. La monotonie de $\mathcal{O}$ se démontre en quelques étapes en appliquant un résultat connu sur une classe particulière des systèmes dynamiques appelés systèmes compétitifs. Ce résultat permet de conclure des relations d'ordre sur les solutions (voir [Smi08]) et par conséquent déduire que $\mathcal{O}$ est décroissant.

L'existence d'une trajectoire T_2-périodique résulte du fait que $\mathcal{O}$ admet un unique point fixe $x_{2,0}^\star \in (0, s_{in} - x_1^r)$. La solution $x^\star(\cdot)$ associée à $D(\cdot)$ défini dans la Proposition 3.3 telle que $x^\star(0) = (x_1^r, x_{2,0}^\star)$ est alors l'unique solution périodique de période $T_2 = T_2^\star$. Nous mettons le résultat de la contraction de l'opérateur $\mathcal{O}$ en conjecture, celle ci étant importante pour étudier la convergence asymptotique des solutions. Comme les instants T_i ne sont pas donnés explicitement, nous sommes amenés à utiliser la théorie des systèmes asymptotiquement périodiques (voir [Zha96]) pour montrer le résultat suivant.

Proposition 3.5. *Pour toute condition initiale $x(0) \in \mathcal{K}(x_1^r)$ il existe $\bar{\sigma} > 0$ tel que la solution $x(\cdot)$ associée à la fonction $D(\cdot)$ faiblement résiliente converge asymptotiquement vers $x^\star(\cdot)$ jusqu'un décalage de temps égale à $\bar{\sigma}$, c'est à dire*

$$\lim_{t \to +\infty} (x(t + \bar{\sigma}) - x^\star(t)) = 0. \tag{1.27}$$

Les simulations numériques présentées sur la Fig. 1.16 illustrent le résultat de convergence précédent.

Le Chapitre 6 de cette thèse est en lien direct avec le chapitre qui le précède. Il est à noter que la fonction $D(\cdot)$ dite faiblement résiliente construite précédemment est considérée comme une loi de commande en boucle ouverte. Nous proposons alors de réécrire cette fonction comme un retour

12. Soit le système contrôlé (1.1) et soit $x(\cdot, x_0)$ son unique solution. On considère l'ensemble fermé $\mathcal{T} \subset \mathbb{R}^n$ appelé *cible*. La fonction appelé *first entry time function* est donnée par

$$R(x_0) := \inf\{t \geq 0\ ;\ x(t, x_0) \in \mathcal{T}\}, \quad x_0 \in \mathbb{R}^n.$$

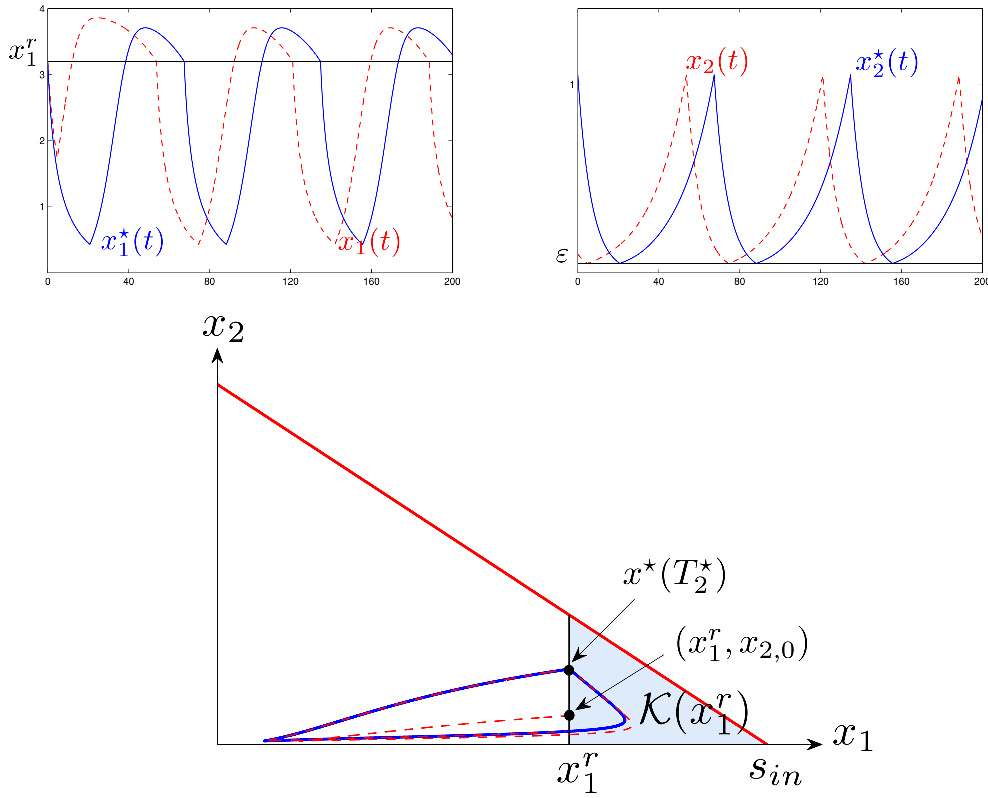


FIGURE 1.16 – *Convergence d'une solution $x(\cdot)$ de (1.26) associée à $D(\cdot)$ défini dans la Proposition 3.3 vers la solution périodique $x^\star(\cdot)$.*

d'état *hybride* associé à un signal de trois états. Un retour d'état classique (feedback) n'est pas possible car on doit conserver la "mémoire" de l'entrée ou de la sortie de l'ensemble $\mathcal{E}$, ce qui peut s'effectuer à l'aide d'un automate à 3 états. L'un des principaux avantages de cette autre synthèse est qu'elle ne nécessite pas une connaissance précise des caractéristiques de croissance de l'espèce envahissante. En pratique, on peut simplement appliquer ce contrôle même en l'absence de cette espèce, ce qui garantit une robustesse des performances face aux éventuelles invasions futures. Nous avons également étudié en fonction des paramètres (D_m, D_M et ε) le temps passé dans l'ensemble désirable $\mathcal{K}(x_1^r)$ et aussi la productivité de l'espèce d'intérêt (espèce native). On observe numériquement que le pourcentage de temps passé dans $\mathcal{K}(x_1^r)$ et également la productivité moyenne sont d'autant plus élevés que l'on ramène la trajectoire suffisamment proche de l'origine (vers l'ensemble $\mathcal{E}$ avec D_M), ce qui est possible avec un ϵ suffisamment petit.

Plan de la thèse

Cette thèse est composé de cinq chapitres présentés sous forme d'articles:

- Le Chapitre 2 fait l'objet d'un article accepté dans le journal *Mathematical Control and Related Fields*.
- Le Chapitre 3 fait l'objet d'un article soumis dans le journal *AUTOMATICA*.
- Le Chapitre 4 est à paraître en ligne dans les *proceedings of the* 58^{th} *IEEE Conference on Decision and Control*.
- Le Chapitre 5 est déposé sur HAL comme une pré-publication.
- Le Chapitre 6 est à paraître en ligne dans les *proceedings of the* 58^{th} *IEEE Conference on Decision and Control*.

Les trois premiers chapitres cités ci-dessus portent sur un même thème concernant l'étude d'un problème de contrôle optimal périodique régi par un système dynamique scalaire, sous contrainte intégrale. Les deux derniers chapitres portent sur la résilience dans le modèle de chemostat en présence d'une espèce envahissante.

Tous les articles cités ci-dessus ont été réalisés en collaboration avec mes directeurs de thèse Alain Rapaport* et Térence Bayen**.

Nous terminons ce travail par une conclusion générale qui résume les principaux résultats obtenus. Quelques perspectives pour les études futures sont également émises. En Annexe, nous présentons quelques résultats classiques en théorie du contrôle optimal, en particulier le Principe de Maximum de Pontryagin.

Pour faciliter la lecture des articles, leurs bibliographies ont été regroupées dans une bibliographie commune en fin du manuscrit.

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Chapitre 2

Optimal periodic control for scalar dynamics under integral constraint on the input

Ce travail est accepté dans le journal *Mathematical Control and Related Fields.*

Contents

1 Introduction

In many applications, the control of dynamical models allows to drive the state x of a system to an operating point, typically a steady state $\bar{x}$ which is an equilibrium point of the dynamics under a constant control $\bar{u}$. When a criterion of performance is associated with the state of the system, it may happen that a periodic trajectory near the steady state gives a better averaged performance than at steady state. But such a gain in the performance could be at the price of higher effort (or cost) on the control variable. The objective of the present work is to investigate the possibility of improving the performance of a steady state with periodic solutions, while keeping the same control effort over each time period. We consider in this work that this effort is measured by the integral of the control $u(\cdot)$ over a period. Keeping the same effort consists then in imposing that the averaged control over a periodic solution is equal to the control $\bar{u}$ at steady state. For this purpose, we formulate an optimal control problem over periodic solutions, under an integral constraint on the control. Periodic optimal control has already been investigated in the literature, mainly under the consideration that solutions are sought near a steady state optimizing the criterion among stationary solutions. In particular, the so-called π-criterion characterizes the existence of "best" periods. It consists first in determining an optimal steady state among constant controls, and then in checking on a linear-quadratic approximation if there exists a frequency of a periodic signal near the nominal constant one that could improve the cost (see [BFG73, BG80]). For instance, in [AL87, AL89, HLPS93], this method has been applied on the chemostat model, and it has been shown that its productivity can be improved with a periodic control when there is a delay in the dynamics. However, there are relatively few theoretical works about global optimality of periodic controls (apart from [Maf74] for the characterization of the value function under quite strong assumptions). Most of the existing works deal with local necessary conditions ([GLR74, Gil77]), second order conditions ([BLM74, SE84, WS90]) or approximations techniques ([ESC87, AL88, BV14]). In [BV14] for instance, a local analysis is conducted in the context of age-structured system showing how to improve locally the cost function by considering periodic controls versus constant ones (but no integral constraint on the control is considered). It has to be underlined that, in our approach, we do not have to consider that the steady state is optimizing the criterion among all stationary solutions of the system (the optimal steady-state control does not necessarily satisfy the integral constraint). To our knowledge, integral constraint on the control has not been yet considered in problems of determining optimal periodic trajectories. Therefore, our objective is some what different than what has been described above.

In applications for which the control variable is a flow rate of matter (such as in continuously fed reactors for instance [HLRS17]), this constraint amounts to consider that a given quantity of matter is available for each period of time, and the problem is then to determine how to deliver this matter during this period (*i.e.*, at a constant flow rate or not?), maintaining a periodic operation over the future times and maximizing the production or the quality of a product over each period. The present problem has been mainly motivated by the modelling of exploited populations of stock (or density) x, see, *e.g.*, [Cla10], for which the control variable u is the harvesting effort (for instance the number of fishermen boats on a lake). In our setting, for a given steady state $\bar{x}$ and its associated constant control $\bar{u}$, we consider the set of T-periodic trajectories with periodic controls having $\bar{u}$ as average. We say that a *over-yielding* occurs when the averaged utility of the stock $x(\cdot)$ of a T-periodic solution is larger than the utility of the stock $\bar{x}$. Let us finally mention [Ide06, IW06] where periodic inputs are studied in the context of population biology and fisheries management, but with different objectives (no optimization and no such integral constraint are considered).

To our knowledge, this problem has not been yet addressed theoretically in the literature. From a mathematical view point, the integral constraint on the input brings two main difficulties:

1. the existence of non-constant periodic trajectories with a control satisfying the integral constraint,
2. the characterization of an optimal control under both constraints of periodicity of the trajectory and the integral constraint on the input,

that we propose to tackle here for scalar dynamics in general framework.

The paper is organized as follows. In Section 2, we formulate the problem and give a precise definition of over-yielding. We then provide assumptions on the dynamics and the cost function that guarantee or prevent over-yielding. In particular, we show that convexity is playing an important role. In section 3, we synthesize optimal periodic controls (in particular non constant ones) improving the cost function compared to steady-state (see Theorem 3.6). In Section 4, we show how to relax the assumptions of Section 2 that are required on an invariant domain (a, b) of the dynamics, when these ones are fulfilled only in a neighborhood of $\bar{x}$. This leads us to give a result similar to the one of Section 3 but for restrictive values of the period T. Finally, we illustrate the results of Section 3-4 in Section 5 in the context of sustainable resource management (see, *e.g.*, [Cla10]). We study the impact on the stock of non-constant periodic inputs (harvesting efforts) but with the same average value, and determine the worst-case scenarios with respect to a given utility of the stock.

2 Existence of over-yielding

Given two functions $f, g : \mathbb{R} \to \mathbb{R}$ of class C^1, we consider the control system

$$\dot{x} = f(x) + ug(x), \tag{2.1}$$

where u is a control variable taking values in $[-1, 1]$. We suppose that the system satisfies the following hypotheses:

(H1) There exists $(a, b) \in \mathbb{R}^2$ with $a < b$ such that g is positive on the interval $I := (a, b)$ with

$$f(a) - g(a) = 0 \quad \text{and} \quad f(b) + g(b) = 0.$$

(H2) One has $f - g < 0$ and $f + g > 0$ on I.

Remark 2.1. *Hypothesis (H1) implies that the interval I is invariant by* (2.1) *whereas Hypothesis (H2) is related to controllability properties of* (2.1) *(that will* *be used in the next section for the synthesis of non-constant periodic trajectories). In the rest of the paper, we shall consider initial conditions in I only.*

We define for $x \in I$ the function

$$\psi(x) := -\frac{f(x)}{g(x)}.$$

Notice that Hypotheses (H1)-(H2) imply that one has $\psi(I) \subset [-1, 1]$. Therefore, for any $\bar{x} \in I$, the control value $\bar{u}$ defined as $\bar{u} := \psi(\bar{x})$ is such that $\bar{u} \in [-1, 1]$. Note that any such point $\bar{x}$ is an equilibrium of (2.1) for the constant control $u = \bar{u}$. Throughout the paper, we fix a point $\bar{x} \in I$ as a nominal steady state. In the sequel, we shall consider T-periodic solutions of (2.1), where $T \in \mathbb{R}_+^*$, with a T-periodic control u that satisfies the integral constraint

$$\frac{1}{T}\int_0^T u(t)\,\mathrm{d}t = \bar{u}. \tag{2.2}$$

We then define the set $\mathcal{U}_T$ of admissible controls as

$$\mathcal{U}_T := \{u : [0, +\infty) \to [-1, 1] \text{ s.t. } u \text{ is meas.}, \ \ T\text{-periodic and fulfills (2.2)}\}. \tag{2.3}$$

One has the following property.

Lemma 2.1. *Under Hypothesis* (H1)*, any T-periodic solution x of* (2.1) *in I with $u \in \mathcal{U}_T$ fulfills the property*

$$\int_0^T (\psi(x(t)) - \psi(\bar{x}))\ \mathrm{d}t = 0. \tag{2.4}$$

Proof. On the interval I, the function g is positive and from equation (2.1), we get

$$\int_0^T \frac{\dot{x}(t)}{g(x(t))}\,\mathrm{d}t = -\int_0^T \psi(x(t))\,\mathrm{d}t + \int_0^T u(t)\,\mathrm{d}t.$$

Define the function

$$h(x) := \int_{\bar{x}}^x \frac{\mathrm{d}\xi}{g(\xi)}, \quad x \in I,$$

together with the function $t \mapsto y(t) := h(x(t))$ for $t \in [0, T]$. For any control function u that fulfills the constraint (2.2), one then has

$$y(T) - y(0) = -\int_0^T (\psi(x(t)) - \bar{u})\,\mathrm{d}t,$$

where $\bar{u} = \psi(\bar{x})$. For any T-periodic solution x in I, y is also T-periodic and one obtains the property (2.4). □

We now require the following hypothesis on $\bar{x}$.

($\bar{\mathrm{H}}$) The function ψ satisfies the property.

$$(\psi(x) - \psi(\bar{x}))(x - \bar{x}) > 0, \quad \forall x \in I \setminus \{\bar{x}\}.$$

This hypothesis is related with the asymptotic stability of $\bar{x}$ for the dynamics (2.1) in I with the constant control $\bar{u}$ (as we shall see in Lemma (2.2), see (2.5)). For applications, it sounds also reasonable that the given steady state $\bar{x}$ is a stable equilibrium of the system under constant control.

For convenience, we denote by $t \mapsto x(t, u, x_0)$ the solution of (2.1) with $u \in \mathcal{U}_T$ and taking the value $x_0 \in I$ at time 0. In the following, we shall consider T-periodic solutions with the initial condition $x(0) = \bar{x}$ (*i.e.*, that are such that $x(T, u, \bar{x}) = \bar{x}$ for $u \in \mathcal{U}_T$). We first show that Hypothesis ($\bar{\mathrm{H}}$) guarantees the existence of non-constant such solutions.

Lemma 2.2. *Under Hypotheses* (H1)-($\bar{\mathrm{H}}$), *there exist non-constant T-periodic solutions of* (2.1) *with $x(0) = \bar{x}$ and $u \in \mathcal{U}_T$, for any $T > 0$.*

Proof. Consider the constant control $u = \bar{u}$ and its associated dynamics in I

$$\dot{x} = \bar{f}(x) := g(x)(\bar{u} - \psi(x)) = g(x)(\psi(\bar{x}) - \psi(x)). \tag{2.5}$$

As the function g is positive on I, Hypothesis ($\bar{\mathrm{H}}$) implies that one has $\bar{f} < 0$ on $(\bar{x}, b)$, and $\bar{f} > 0$ on $(a, \bar{x})$. Therefore, one has the properties

$$\begin{aligned} x_0 \in (\bar{x}, b) \quad &\Rightarrow \quad x(T, \bar{u}, x_0) < x_0, \\ x_0 \in (a, \bar{x}) \quad &\Rightarrow \quad x(T, \bar{u}, x_0) > x_0. \end{aligned} \tag{2.6}$$

Consider now any bounded T-periodic measurable function $v : [0, +\infty) \to [-1, 1]$ satisfying

$$\int_0^T v(t)\,\mathrm{d}t = 0,$$

and the control function

$$u_\varepsilon(t) := \bar{u} + \varepsilon v(t),$$

where $\varepsilon \in \mathbb{R}$. Clearly, u_ε satisfies the constraint (2.2) and for ε small enough, one has $u_\varepsilon(t) \in [-1,1]$ for any $t \geq 0$. Define then the function

$$\theta(x_0, \varepsilon) := x(T, u_\varepsilon, x_0) - x_0,$$

for $(x_0, \varepsilon) \in I \times \mathbb{R}$. By the Theorem of continuous dependency of the solutions of ordinary differential equations w.r.t. initial conditions and parameters (see for instance [Per13]), θ is a continuous function. From (2.6), we deduce that

$$\begin{array}{lll} x_0 \in (\bar{x}, b) & \Rightarrow & \theta(x_0, 0) < 0, \\ x_0 \in (a, \bar{x}) & \Rightarrow & \theta(x_0, 0) > 0, \end{array}$$

and by continuity of θ, there exists $\varepsilon \neq 0$, $x_0^+ \in (\bar{x}, b)$ and $x_0^- \in (a, \bar{x})$ such that $\theta(x_0^+, \varepsilon) < 0$ and $\theta(x_0^-, \varepsilon) > 0$. By the Mean Value Theorem, we deduce the existence of $x_0 \in (x_0^-, x_0^+)$ such that $\theta(x_0, \varepsilon) = 0$, that is, the existence of a T-periodic solution x of (2.1) with a non-constant control u that satisfies the constraint (2.2). From Lemma 2.1, such solution satisfies

$$\int_0^T (\psi(x(t)) - \psi(\bar{x}))\, \mathrm{d}t = 0,$$

which implies that the map $t \mapsto \psi(x(t)) - \psi(\bar{x})$ cannot be of constant sign on $[0, T]$. Hypothesis $(\bar{\mathrm{H}})$ implies that $x(t) - \bar{x}$ has to change its sign. Therefore there exists $\bar{t} \in (0, T)$ with $x(\bar{t}) = \bar{x}$ in such a way that the control function $\tilde{u}$ defined by $t \mapsto \tilde{u}(t) := u(t + \bar{t})$ guarantees to have $x(T, \tilde{u}, \bar{x}) = \bar{x}$. □

Now, let $\ell : \mathbb{R} \to \mathbb{R}$ be a function of class C^1 and consider the cost function

$$J_T(u) := \frac{1}{T} \int_0^T \ell(x_u(t))\, \mathrm{d}t, \tag{2.7}$$

where x_u is the unique solution of (2.1) such that $x_u(0) = \bar{x}$, associated with a control $u \in \mathcal{U}_T$. Our aim in this work is to address the question of finding a periodic trajectory with $x(0) = \bar{x}$ that has a lower cost than the constant $\bar{x}$, with a (T-periodic) control of mean value $\bar{u}$. For this purpose, we introduce the following terminology.

Definition 2.1. *Given $T > 0$, we say that* (2.1) *exhibits an over-yielding for the cost* (2.7) *if there exists a T-periodic solution x of* (2.1) *with $x(0) = \bar{x}$ associated with a control $u \in \mathcal{U}_T$ such that $J_T(u) < \ell(\bar{x})$.*

Moreover, we aim to characterize in the next section the strategies realizing the minimum of the criterion (2.7) among such controls. The possibility of having an over-yielding relies on specific assumptions on the cost function and the dynamics, that we now introduce.

(H3) The function $\ell : I \to \mathbb{R}$ is increasing and the function $\gamma := \psi \circ \ell^{-1}$ is strictly convex increasing over $\ell(I)$.

Remark 2.2. *Hypothesis* (H3) *implies Hypothesis* ($\bar{\mathrm{H}}$)*. Therefore, by Lemma 2.2, there exist T-periodic solutions x of* (2.1) *with $x(0) = \bar{x}$ and $u \in \mathcal{U}_T$, that are different of the constant solution $\bar{x}$, when* (H1)-(H2)-(H3) *are fulfilled. Hypothesis* (H3) *also implies that ψ is increasing.*

Proposition 2.1. *If* (H1) *and* (H3) *hold true, any non-constant T-periodic solution x of* (2.1) *with $x(0) = \bar{x}$ and $u \in \mathcal{U}_T$ satisfies $J_T(u) < \ell(\bar{x})$.*

Proof. Consider a T-periodic solution x with $x(0) = \bar{x}$ associated with a control in $\mathcal{U}_T$. From Lemma 2.1, equality (2.4) is satisfied and we deduce

$$\int_0^T (\gamma(\ell(x(t))) - \gamma(\ell(\bar{x})))\,\mathrm{d}t = 0.$$

For a non-constant solution, we find by Jensen's inequality

$$\gamma\left(\frac{1}{T}\int_0^T \ell(x(t))\,\mathrm{d}t\right) < \frac{1}{T}\int_0^T \gamma(\ell(x(t)))\,\mathrm{d}t = \gamma(\ell(\bar{x})).$$

Since γ is increasing over $\ell(I)$ with, we obtain

$$J_T(u) = \frac{1}{T}\int_0^T \ell(x(t))\,\mathrm{d}t < \ell(\bar{x}).$$

□

Remark 2.3. (i) *The result of Proposition 2.1 applies in the simple case where $\ell(x) = x$ and ψ is strictly convex and increasing over I.*
(ii) *If ψ is strictly convex and increasing over I and ℓ is strictly concave increasing over I, the result of Proposition 2.1 also holds true (by a similar reasoning).*

We now provide sufficient conditions for preventing any over-yielding.

(H4) There exists a continuous function $\bar{\psi}$ such that

(i) $\bar{\psi} \geq \psi$ on I with $\bar{\psi}(\bar{x}) = \psi(\bar{x})$,

(ii) the function $\bar{\gamma} := \bar{\psi} \circ \ell^{-1}$ is concave increasing on $\ell(I)$.

Proposition 2.2. *If* (H1) *and* (H4) *hold true then no over-yielding is possible.*

Proof. We suppose by contradiction that there exists a periodic solution x associated with a control $u \in \mathcal{U}_T$ such that

$$J_T(u) = \frac{1}{T}\int_0^T \ell(x(t))\,\mathrm{d}t < \ell(\bar{x}),$$

The function $\bar{\gamma}$ being increasing on $\ell(I)$, we have

$$\bar{\gamma}\Big(\frac{1}{T}\int_0^T \ell(x(t))\,\mathrm{d}t\Big) < \bar{\gamma}(\ell(\bar{x})) = \bar{\psi}(\bar{x}) = \psi(\bar{x}). \tag{2.8}$$

Using Jensen's inequality for $\bar{\gamma}$, we can write

$$\bar{\gamma}\Big(\frac{1}{T}\int_0^T \ell(x(t))\,\mathrm{d}t\Big) \geq \frac{1}{T}\int_0^T \bar{\gamma}(\ell(x(t)))\,\mathrm{d}t. \tag{2.9}$$

As one has $\bar{\psi} = \bar{\gamma} \circ \ell \geq \psi$ over , we get

$$\frac{1}{T}\int_0^T \bar{\gamma}(\ell(x(t)))\,\mathrm{d}t \geq \frac{1}{T}\int_0^T \psi(x(t))\,\mathrm{d}t. \tag{2.10}$$

Combining inequalities (2.8), (2.9), (2.10), we obtain

$$\psi(\bar{x}) > \frac{1}{T}\int_0^T \psi(x(t))\,\mathrm{d}t,$$

which is a contradiction with the equality (2.4) given by Lemma 2.1. □

Remark 2.4. (i) *Thanks to the previous proposition, if $\ell(x) = x$ for $x \in \mathbb{R}$ and ψ is strictly concave, then no over-yielding is possible. In the same way, if ℓ is increasing on I and γ strictly concave increasing over $\ell(I)$, then the same conclusion follows.*
(ii) *Under hypotheses (H1)-(H3), we say that an over yielding is systematic (which means that it exists for any $T > 0$, see Proposition 2.1).*

3 Determination of optimal periodic solutions

In this Section, we assume that Hypotheses (H1)-(H2)-(H3) hold true, so that we know that over-yielding is possible (actually, it is systematic according to Proposition 2.1). For a given $T > 0$, we shall say that a solution x of (2.1) is *T-admissible* if it is T-periodic with $x(0) = \bar{x}$ and $u \in \mathcal{U}_T$. We reformulate the control constraint (2.2) by considering the augmented dynamics

$$\begin{cases} \dot{x} &= f(x) + ug(x), \\ \dot{y} &= u, \end{cases} \tag{2.11}$$

together with the boundary conditions:

$$(x(0), y(0)) = (\bar{x}, 0) \quad \text{and} \quad (x(T), y(T)) = (\bar{x}, \bar{u}T). \tag{2.12}$$

The optimal control problem can be then stated as follows

$$\inf_{u \in \mathcal{U}} \int_0^T \ell(x(t))\,\mathrm{d}t \quad \text{s.t. } (x, y) \text{ satisfies } (2.11) - (2.12), \tag{2.13}$$

where $\mathcal{U}$ denotes the set of measurable control functions u over $[0, T]$ taking values in $[-1, 1]$. Note that Problem (2.13) admits a solution by classical existence results. Indeed, hypotheses (H1)-(H2)-(H3) imply that there exist trajectories of (2.11) satisfying (2.12). Since the system is affine w.r.t. the control and ℓ is continuous, the existence of an optimal control follows by Filippov's existence theorem [Ces12].

3.1 Application of the Pontryagin Maximum Principle

We derive necessary optimality conditions using the Pontryagin Maximum Principle [LPM64]. Let $H : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the Hamiltonian associated with (2.13):

$$H = H(x, y, \lambda_x, \lambda_y, \lambda_0, u) = \lambda_0 \ell(x) + \lambda_x f(x) + u(\lambda_x g(x) + \lambda_y),$$

where $\lambda := (\lambda_x, \lambda_y)$ denotes the adjoint vector. Let $u \in \mathcal{U}$ be an optimal control and (x, y) a solution of (2.11)-(2.12) associated with u. Then, there exists a scalar $\lambda_0 \leq 0$ and an absolutely continuous map $\lambda : [0,T] \to \mathbb{R}^2$ satisfying the adjoint equation

$$\begin{cases} \dot{\lambda}_x = -\lambda_0 \ell'(x(t)) - \lambda_x(f'(x(t)) + u(t)g'(x(t))), \\ \dot{\lambda}_y = 0, \end{cases} \tag{2.14}$$

for a.e. $t \in [0,T]$. Moreover, $(\lambda_0, \lambda) \neq 0$ and the Hamiltonian condition writes

$$u(t) \in \arg \max_{\omega \in [-1,1]} H(x(t), \lambda(t), \lambda_0, \omega) \quad \text{a.e. } t \in [0,T]. \tag{2.15}$$

Since the dynamics is affine w.r.t. u, the switching function

$$t \mapsto \phi(t) := \lambda_x(t) g(x(t)) + \lambda_y,$$

provides the following expression of the control u (thanks to (2.15)):

$$\begin{cases} \phi(t) > 0 & \Rightarrow \quad u(t) = 1, \\ \phi(t) < 0 & \Rightarrow \quad u(t) = -1, \\ \phi(t) = 0 & \Rightarrow \quad u(t) \in [-1,1]. \end{cases} \tag{2.16}$$

Moreover, if we differentiate ϕ w.r.t t, we find that for $t \in [0,T]$

$$\dot{\phi}(t) = \lambda_x(t)[f(x(t))g'(x(t)) - f'(x(t))g(x(t))] - \lambda_0 \ell'(x(t))g(x(t)).$$

An extremal trajectory is a quadruple $(x, \lambda, \lambda_0, u)$ where (x, λ) satisfies the state-adjoint equations and u the Hamiltonian condition (2.15). We recall that a singular arc occurs if ϕ vanishes on some time interval $[t_1, t_2]$ with $t_1 < t_2$, and a switching time $t_s \in (0,T)$ is such that an extremal control u is non-constant in any neighborhood of t_s (which implies that $\phi(t_s) = 0$). It is also worth to mention that from Hypothesis (H2), when $\phi > 0$, resp. $\phi < 0$, then x is increasing, resp. decreasing.

Lemma 3.1. *Under Hypotheses* (H1)-(H2)-(H3), *there is no abnormal extremal trajectory, i.e.,* $\lambda_0 \neq 0$.

Proof. If $\lambda_0 = 0$, then λ_x cannot vanish from the adjoint equation. Otherwise λ_x would be zero over $[0,T]$ and the switching function would be constant equal to λ_y. Since λ_y cannot be simultaneously equal to 0, ϕ would be of constant sign over $[0,T]$ implying that $u=1$ or $u=-1$ over $[0,T]$ and a contradiction with the periodicity of $x(\cdot)$ (recall that $f+g>0$ and $f-g<0$ over I). As a consequence, λ_x is of constant sign. Now, since $\lambda_0=0$, one has

$$\dot{\phi}(t)=\lambda_x(t)g(x(t))^2\psi'(x(t)), \qquad t\in[0,T].$$

We deduce that $\dot{\phi}$ is of constant sign (recall that $\psi'>0$), hence ϕ is monotone. Consequently, the extremal trajectory has at most one switching point. Thus, one has $x(t)>\bar{x}$ for any time $t\in(0,T)$ implying a contradiction with (2.4). If $x(t)<\bar{x}$ for any time $t\in(0,T)$, we conclude in the same way. □

Without any loss of generality, we may assume that $\lambda_0=-1$.

Remark 3.1. *Considering T-periodic optimal solutions in I without requiring the initial condition $x(0)=\bar{x}$, but only $x(T)=x(0)$ provides the transversality condition $\lambda_x(T)=\lambda_x(0)$. However, Lemma 2.1 and Hypothesis (H3) (or simply ($\bar{\mathrm{H}}$)) imply that any T-periodic optimal solution $x(\cdot)$ in I has to pass by $\bar{x}$. Therefore, we can impose $x(0)=\bar{x}$ without any loss of generality, and deduce that $\lambda_x(\cdot)$ is necessarily T-periodic (even though we shall not use this property in the following).*

3.2 Properties of switching times

Let us denote by x_m and x_M the minimum and maximum on $[0,T]$ of a T-admissible solution x. Note that for any time $t\in(0,T)$ such that $x(t)\in\{x_m,x_M\}$, then one has $\phi(t)=0$. Indeed, otherwise one would have $\phi(t)>0$ or $\phi(t)<0$. Suppose for instance that $\phi(t)>0$. From (2.16), the control u would be equal to 1 in a neighborhood of t, and thus, from (H2), we would have a contradiction with the fact that x_M is the maximum of x. We proceed in the same way if $\phi(t)<0$.

Proposition 3.1. *Under Hypotheses* (H1)-(H2)-(H3)*, any extremal satisfies the following properties.*

1. *At any switching time $t_s\in(0,T)$, one has $x(t_s)\in\{x_m,x_M\}$.*
2. *It has no singular arc.*

Proof. Let t_1, t_2 in $[0, T]$ be such that $x(t_1) = x_m$ and $x(t_2) = x_M$ with x_m, x_M in I. We deduce that $\lambda_x(t_1)g(x_m) = \lambda_x(t_2)g(x_M) = -\lambda_y$. Now, since H is conserved along any extremal trajectory (see for instance [Ces12]), one has

$$H = -\ell(x_M) - \lambda_y \frac{f(x_M)}{g(x_M)} = -\ell(x_m) - \lambda_y \frac{f(x_m)}{g(x_m)},$$

implying that (recall that $\gamma = \psi \circ \ell^{-1}$)

$$\frac{1}{\lambda_y} = \frac{\psi(x_M) - \psi(x_m)}{\ell(x_M) - \ell(x_m)} = \frac{\gamma(\ell(x_M)) - \gamma(\ell(x_m))}{\ell(x_M) - \ell(x_m)}, \tag{2.17}$$

As γ is increasing over $\ell(I)$, one has $\lambda_y > 0$. Suppose now that t_s is a switching time such that $x(t_s) \in (x_m, x_M)$. Using a similar computation as above, we find that

$$\frac{1}{\lambda_y} = \frac{\psi(x_M) - \psi(x(t_s))}{\ell(x_M) - \ell(x(t_s))} = \frac{\gamma(\ell(x_M)) - \gamma(\ell(x(t_s)))}{\ell(x_M) - \ell(x(t_s))}. \tag{2.18}$$

Since γ and ℓ are respectively strictly convex and increasing on $[x_m, x_M]$, (2.17) and (2.18) imply a contradiction, thus $x(t_s) \in \{x_m, x_M\}$ as was to be proved.

Suppose now by a contradiction that there exists a time interval $[t_1, t_2]$ such that $\phi(t) = \dot{\phi}(t) = 0$ for $t \in [t_1, t_2]$. Combining $\phi = \dot{\phi} = 0$ over $[t_1, t_2]$, one finds that

$$\begin{aligned} & \ell'(x(t)) - \lambda_y \psi'(x(t)) = 0, && \forall t \in [t_1, t_2], \\ \Rightarrow \quad & 1 - \lambda_y \gamma'(\ell(x(t))) = 0, && \forall t \in [t_1, t_2] \text{ (recall that } \psi = \gamma \circ \ell), \\ \Rightarrow \quad & \tfrac{1}{\lambda_y} = \gamma'(\ell(x(t))), && \forall t \in [t_1, t_2], \end{aligned}$$

Now, since the extremities of the singular arc t_1 and t_2 must be switching times, one must have $x(t_1)$, $x(t_2)$ in $\{x_m, x_M\}$. Suppose for instance that $x(t_1) = x_m$. One then gets $\frac{1}{\lambda_y} = \gamma'(\ell(x_m))$ which is a contradiction with (2.17) (since γ is strictly convex) and similarly at $t = t_2$. This completes the proof. □

Remark 3.2. (i) *At this stage, we have thus proved that optimal trajectories are of bang-bang type (i.e. they are concatenations of arcs with $u = \pm 1$) such that at each switching time t_s one has $x(t_s) \in \{x_m, x_M\}$. One can show that the number*

of switching times is finite (by doing a similar reasoning as for the exclusion of singular arcs).
(ii) *Moreover, this number is necessarily even. Indeed, let $x(\cdot)$ be a T-admissible solution of* (2.1) *associated with a control $u \in \mathcal{U}_T$ having $2n+1$ switching times over $[0,T]$ with $n > 0$. Note that $n = 0$ is impossible since the map $t \mapsto x(t) - \bar{x}$ has to change its sign over $[0,T]$. Hence, it has to be equal at least once to $\bar{x}$ on the interval $(0,T)$, say at a time $\bar{t}$, to satisfy* (2.4) *which invalidates $n = 0$. Finally, observe that the sign of $\dot{x}(0^+)$ and $\dot{x}(T^-)$, with an odd number of switches, are necessarily distinct. It follows that the sign of $\dot{x}(T^-)$ and $\dot{x}(T^+)$ are also distinct. From the initial condition $x(\bar{t}) = \bar{x}$, the T-periodic solution over $(\bar{t}, T+\bar{t})$ switches then at time $t = T$, which belongs to the interval $(\bar{t}, T+\bar{t})$. Since $x(T) = \bar{x}$, we have a contradiction with point 1 of Proposition 3.1. Hence the number of switches is even.*

We focus now on extremal trajectories with two switches.

3.3 Trajectories with two switches

For a given $T > 0$, we consider trajectories $t \mapsto x(t)$ solutions of (2.1) on $[0,T]$ with $x(0) = \bar{x}$ and associated with a control u defined by two switching times t_1, t_2 with $0 < t_1 < t_2 < T$:

$$u(t) = \left| \begin{array}{rl} 1, & t \in [0, t_1), \\ -1, & t \in [t_1, t_2), \\ 1, & t \in [t_2, T]. \end{array} \right. \tag{2.19}$$

These trajectories, that we shall call $B_+B_-B_+$ *trajectories*, will play an important role in the following. Note that under Hypotheses (H1)-(H2) a $B_+B_-B_+$ trajectory is characterized uniquely by its maximal and minimal values $x_M = x(t_1)$ and $x_m = x(t_2)$ in I. For convenience, we define on the interval I the function

$$\eta(x) := \frac{1}{f(x)+g(x)} - \frac{1}{f(x)-g(x)}.$$

From Hypothesis (H2), note that η is C^1 and positive function on I.

Lemma 3.2. *Under Hypotheses* (H1)-(H2)*, if a $B_+B_-B_+$ trajectory is T-periodic, then the pair (x_m, x_M) satisfies*

$$\int_{x_m}^{x_M} \eta(x)\,\mathrm{d}x = T. \tag{2.20}$$

Moreover, if the corresponding control satisfies (2.2) *then the pair (x_m, x_M) satisfies*

$$\int_{x_m}^{x_M} \eta(x)\psi(x)\,\mathrm{d}x = \bar{u}T. \tag{2.21}$$

Proof. For $t \in [0, t_1) \cup [t_2, T)$, one has $\dot{x} = f(x) + g(x) > 0$ and one can write

$$t_1 = \int_{\bar{x}}^{x_M} \frac{\mathrm{d}x}{f(x)+g(x)}, \quad T - t_2 = \int_{x_m}^{x(T)} \frac{\mathrm{d}x}{f(x)+g(x)}.$$

Similarly for $t \in [t_1, t_2)$, one has $\dot{x} = f(x) - g(x) < 0$ and

$$t_2 - t_1 = -\int_{x_m}^{x_M} \frac{\mathrm{d}x}{f(x)-g(x)}.$$

One then obtains

$$T = \int_{x_m}^{x(T)} \frac{\mathrm{d}x}{f(x)+g(x)} - \int_{x_m}^{x_M} \frac{\mathrm{d}x}{f(x)-g(x)} + \int_{\bar{x}}^{x_M} \frac{\mathrm{d}x}{f(x)+g(x)},$$

and for a T-periodic solution, $x(T) = \bar{x}$ gives exactly the property (2.20). Proceeding with the same decomposition of the interval $[0, T]$, one can write

$$\int_0^T u(t)\,\mathrm{d}t = \int_{\bar{x}}^{x_M} \frac{\mathrm{d}x}{f(x)+g(x)} - \int_{x_M}^{x_m} \frac{\mathrm{d}x}{f(x)-g(x)} + \int_{x_m}^{\bar{x}} \frac{\mathrm{d}x}{f(x)+g(x)},$$

which gives the quality

$$\int_{x_m}^{x_M} \left(\frac{1}{f(x)+g(x)} + \frac{1}{f(x)-g(x)} \right) \mathrm{d}x = \bar{u}T,$$

when u fulfills (2.2). Finally, notice that one has

$$\frac{1}{f(x)+g(x)} + \frac{1}{f(x)-g(x)} = \eta(x)\psi(x),$$

for $x \in I$, and thus property (2.21) is satisfied. □

We first analyze the possibilities of satisfying the integral condition (2.20).

Lemma 3.3. *Under Hypotheses* (H1)-(H2)*, for any $T > 0$ there exists a unique function $\beta_T : [a,b] \to [a,b]$ that satisfies $\beta_T(\alpha) > \alpha$ for any $\alpha \in I$ and*

$$\int_{\alpha}^{\beta_T(\alpha)} \eta(x)\,\mathrm{d}x = T, \quad \alpha \in I.$$

Moreover β_T is of class C^1, increasing and bijective from $[a,b]$ to $[a,b]$.

Proof. The function $f+g$ is of class C^1 and positive on I with $(f+g)(b) = 0$. Thus, it is easy to see that $K_+ := -\min_{x\in[a,b]}(f+g)'(x) > 0$. It follows that one has the inequality $(f+g)(x) \le K_+(b-x)$ for any $x \in I$. As the function η satisfies

$$\eta(x) > \frac{1}{f(x)+g(x)} \ge \frac{1}{K_+(b-x)} > 0, \quad x \in I,$$

one deduces that the map

$$\chi : (\xi_-, \xi_+) \mapsto \chi(\xi_-, \xi_+) := \int_{\xi_-}^{\xi_+} \eta(x)\,\mathrm{d}x,$$

is such that for any $\alpha \in I$, $\chi(\alpha, \cdot)$ is of class C^1, increasing with $\chi(\alpha,\alpha) = 0$ and $\chi(\alpha, b) = +\infty$. By the Implicit Function Theorem, there exists a unique map $\beta_T : I \mapsto I$ of class C^1, such that $\chi(\alpha, \beta_T(\alpha)) = T$ for any $\alpha \in I$. Moreover, one has

$$\beta_T'(\alpha) = \frac{\eta(\alpha)}{\eta(\beta_T(\alpha))} > 0, \quad \alpha \in I.$$

The function β_T is thus increasing, and then admits limits at the points a_+ and b_-. Therefore one has $\beta_T(a_+) := \lim_{\alpha\to a_+} \beta_T(\alpha) \ge a$ and $\beta_T(b_-) := \lim_{\alpha\to b_-} \beta_T(\alpha) \le b$ that verify $\chi(a, \beta_T(a_+)) = T$ and $\chi(b, \beta_T(b_-)) = T$, since χ is continuous. As previously, $f - g < 0$ on I with $(f-g)(a) = 0$ implies that $K_- := -\min_{x\in[a,b]}(f-g)'(x) > 0$. It follows that $(f-g)(x) \ge K_-(a-x)$ for any $x \in I$. Thus, we deduce that

$$\eta(x) > -\frac{1}{f(x)-g(x)} \ge \frac{1}{K_-(x-a)} > 0, \quad x \in I.$$

If $\beta_T(a_+) > a$, one should then have $\chi(a, \beta_T(a_+)) = +\infty$ which is not possible since one has $\chi(\alpha, \beta_T(\alpha)) = T$ for any $\alpha \in I$. So, one has $\beta_T(a_+) = a$.

As the function η is positive on I, one also has $\beta_T(\alpha) > \alpha$ for any $\alpha \in I$, and we deduce that $\beta_T(b_-) = b$. This proves that β_T can be extended to a one-to-one mapping from $[a, b]$ to $[a, b]$. □

We are now ready to show that there exists a unique $B_+B_-B_+$ trajectory that satisfies both integral conditions (2.20) and (2.21).

Proposition 3.2. *Under Hypotheses* (H1)-(H2)-($\bar{\mathrm{H}}$)*, there exists a unique pair* $(x_m, x_M) \in I^2$ *satisfying* (2.20)-(2.21)*, and one has* $x_m < \bar{x} < x_M$*.*

Proof. From Lemma 3.3, condition (2.20) implies to have $x_M = \beta_T(x_m)$. We thus have simply to show the uniqueness of x_m for the condition (2.21) to be fulfilled. Consider the function $F : [a, b] \to \mathbb{R}$ defined by

$$F(\alpha) := \int_{\alpha}^{\beta_T(\alpha)} \eta(x)(\psi(x) - \psi(\bar{x}))\,\mathrm{d}x, \tag{2.22}$$

and notice that conditions (2.20) and (2.21) are both satisfied exactly when $F(x_m) = 0$. From Hypothesis ($\bar{\mathrm{H}}$) and the properties satisfied by the function β_T (see Lemma 3.3), one has $F(\alpha) > 0$ for any $\alpha \in [\bar{x}, b)$, and $F(\alpha) < 0$ for any $\alpha \in (a, \beta_T^{-1}(\bar{x})]$. By the Mean Value Theorem, there exists $x_m \in (\beta_T^{-1}(\bar{x}), \bar{x})$ such that $F(x_m) = 0$. Moreover, one has

$$F'(\alpha) = \eta(\beta_T(\alpha))\left(\psi(\beta_T(\alpha)) - \psi(\bar{x})\right)\beta_T'(\alpha) - \eta(\alpha)\left(\psi(\alpha)) - \psi(\bar{x})\right).$$

As β_T is increasing and ψ satisfies ($\bar{\mathrm{H}}$), we obtain $F'(\alpha) > 0$ for any $\alpha < \bar{x}$ with $\beta_T(\alpha) > \bar{x}$, that is exactly for $\alpha \in (\beta_T^{-1}(\bar{x}), \bar{x})$, and we conclude about the existence and uniqueness of x_m, x_M in I, with $x_m < \bar{x}$ and $x_M > \bar{x}$. □

Remark 3.3. *The existence and uniqueness of a $T-$admissible $B_+B_-B_+$ trajectory is a straightforward consequence of Lemma 3.2 and Proposition 3.2. Indeed, under Hypotheses* (H1)-(H2)-($\bar{\mathrm{H}}$)*, Proposition 3.2 allows to uniquely define a pair* (x_m, x_M) *satisfying* (2.20)-(2.21)*. Consider now a solution $x(\cdot)$ of* (2.1) *such that $x(0) = \bar{x}$ which is such that $u = 1$ until $x(\cdot)$ reaches x_M, say at a time t_1 and then $u = -1$ from t_1 until the first time $t_2 > t_1$ such that $x(t_2) = x_m$, and finally $u = 1$ until $x(\cdot)$ reaches $\bar{x}$. For any $T > 0$, this construction defines a unique $B_+B_-B_+$ trajectory that is T-admissible, thanks to* (2.20)-(2.21)*.*

It is also worth to mention that x_m and x_M depend on the period T. In the next Lemma, we provide properties of x_m and x_M as functions of T.

Lemma 3.4. *Under Hypotheses* (H1)-(H2)-($\bar{\text{H}}$), *the functions* $T \mapsto x_m(T)$ *and* $T \mapsto x_M(T)$ *are continuously differentiable, and respectively decreasing and increasing. Moreover, one has*

$$\lim_{T\to+\infty} x_m(T) = a \quad \text{and} \quad \lim_{T\to+\infty} x_M(T) = b. \tag{2.23}$$

Proof. For each $T > 0$, we know from Proposition 3.2 that there exists a unique pair $(x_m(T), x_M(T)) \in I^2$ satisfying (2.20)-(2.21). By the Implicit Function Theorem, x_m and x_M are are continuously differentiable w.r.t. T. Let us denote by ${x_m}'$, ${x_M}'$ the derivatives of x_m and x_M w.r.t. T. Differentiating (2.20)-(2.21) w.r.t. T then yields

$$\underbrace{\begin{bmatrix} \eta(x_M(T)) & -\eta(x_m(T)) \\ \eta(x_M(T))\psi(x_M(T)) & -\eta(x_m(T))\psi(x_m(T)) \end{bmatrix}}_{X(T)} \begin{bmatrix} {x_M}'(T) \\ {x_m}'(T) \end{bmatrix} = \begin{bmatrix} 1 \\ \psi(\bar{x}) \end{bmatrix},$$

where $\det(X(T)) := \eta(x_M(T))\eta(x_m(T))\left(\psi(x_M(T)) - \psi(x_m(T))\right) > 0$. Then ${x_M}'(T)$, ${x_m}'(T)$ are given by the expressions

$$\begin{cases} {x_M}'(T) &= \dfrac{\eta(x_m(T))\left(\psi(\bar{x}) - \psi(x_m(T))\right)}{\det(X(T))} > 0, \\ {x_m}'(T) &= \dfrac{\eta(x_M(T))\left(\psi(\bar{x}) - \psi(x_M(T))\right)}{\det(X(T))} < 0. \end{cases}$$

From (2.20) and (2.21), one has

$$\frac{T}{2}(\bar{u}+1) = \int_{x_m(T)}^{x_M(T)} \frac{\mathrm{d}x}{f(x)+g(x)} < \int_a^{x_M(T)} \frac{\mathrm{d}x}{f(x)+g(x)}.$$

Taking the limit when T tends to $+\infty$ in both side of this inequality, one obtains $\lim_{T\to+\infty} x_M(T) = b$. Similarly one can prove that $\lim_{T\to+\infty} x_m(T) = a$. □

3.4 Optimal solutions

According to Proposition 3.2, for any $T > 0$, we have seen that there is a unique $B_+B_-B_+$ trajectory $\hat{x}_T(\cdot)$ that is T-admissible, generated by a control that we shall denote $\hat{u}_T$. Moreover, there exists a unique $\bar{t} \in (0, T)$ such that $\hat{x}_T(\bar{t}) = \bar{x}$. Therefore, there are exactly two T-admissible solutions $\hat{x}_T(\cdot)$, $\check{x}_T(\cdot)$ with two switches, given by $\hat{u}_T$ and $\check{u}_T$ with

$$\check{u}_T(t) := \hat{u}_T(t + \bar{t}), \quad t \geq 0,$$

which have the same cost. Similarly, we denote by $B_-B_+B_-$ the trajectory $\check{x}_T$. We now study the monotonicity of the cost $J_T(\hat{u}_T)$ with respect to T. This property is crucial for the optimal synthesis (Theorem 3.6) and relies on the convexity assumptions on the data.

Lemma 3.5. *Under Hypotheses* (H1)-(H2)-(H3), *one has*

$$S > T > 0 \Rightarrow J_S(\hat{u}_S) < J_T(\hat{u}_T).$$

Proof. Following (2.19), we denote by t_1 and t_2 the two successive instants of $(0, T)$ for which one has $\hat{u}_T = +1$ over $[0, t_1) \cup [t_2, T]$ and $\hat{u}_T = -1$ over $[t_1, t_2)$. In the same way, we define s_1, s_2 as the two successive instants of $(0, S)$ such that one has $\hat{u}_S = +1$ over $[0, s_1) \cup [s_2, T]$ and $\hat{u}_S = -1$ over $[s_1, s_2)$. Let us also denote by x, y the solutions of (2.1) corresponding to $\hat{u}_T$ and $\hat{u}_S$ respectively and set $x_M := x(t_1)$, $x_m := x(t_2)$, $y_M := y(s_1)$, $y_m := y(s_2)$.

From Lemma 3.4, one has $x_M < y_M$, $x_m > y_m$, $t_1 < s_1$, and $t_2 < s_2$. So, we introduce a E defined by

$$E := \{s \in [0, S] \; ; \; y(s) > x_M \text{ or } y(s) < x_m\},$$

together with a function $\varphi : [0, T] \to [0, S] \backslash E$ by

$$\varphi(t) := \begin{cases} t & \text{if} \quad t \in [0, t_1), \\ t + \delta_1 & \text{if} \quad t \in [t_1, t_2), \\ t + \delta_1 + \delta_2 & \text{if} \quad t \in [t_2, T], \end{cases}$$

where δ_1, resp. δ_2 is the time spent by y over x, resp. below x. They are given by

$$\delta_1 := \mathrm{meas}(\{s \in [0,S]\ ;\ y(s) > x_M\}),\ \delta_2 := \mathrm{meas}(\{s \in [0,S]\ ;\ y(s) < x_m\}).$$

By construction one has $x(t) = y(\varphi(t))$, for $t \in [0,T]$ and φ is bijective, thus $\mathrm{meas}(E) = S - T$. Moreover, for any monotonic function $\rho : I \to \mathbb{R}$ one has

$$\int_0^T \rho(x(t))\,\mathrm{d}t = \int_0^T \rho(y(\varphi(t)))\,\mathrm{d}t = \int_{[0,S]\setminus E} \rho(y(s))\,\mathrm{d}s,$$

by considering the change of variable $s = \varphi(t)$. We then get

$$\int_0^T \ell(x(t))\,\mathrm{d}t = \int_{[0,S]\setminus E} \ell(y(s))\,\mathrm{d}s, \tag{2.24}$$

and

$$\int_0^T \gamma(\ell(x(t)))\,\mathrm{d}t = \int_{[0,S]\setminus E} \gamma(\ell(y(s)))\,\mathrm{d}s.$$

As both controls $\hat{u}_T$ and $\hat{u}_S$ satisfy the constraint (2.4), one has

$$\frac{1}{T}\int_0^T \gamma(\ell(x(t)))\,\mathrm{d}t = \frac{1}{S}\int_0^S \gamma(\ell(y(s)))\,\mathrm{d}s = \bar{u},$$

which implies

$$\frac{1}{S-T}\int_E \gamma(\ell(y(s)))\,\mathrm{d}s = \bar{u}. \tag{2.25}$$

Let us now consider a function $\hat{\gamma} : [\ell(y_m), \ell(y_M)] \to \mathbb{R}$ defined by

$$\hat{\gamma}(\xi) := \left| \begin{array}{ll} \gamma(\ell(x_m)) + \dfrac{\gamma(\ell(x_M)) - \gamma(\ell(x_m))}{\ell(x_M) - \ell(x_m)}(\xi - \ell(x_m)) & \text{for } \xi \in [\ell(x_m), \ell(x_M)], \\ \gamma(\xi) & \text{otherwise,} \end{array} \right.$$

(see Fig. 2.1). First, note that $\hat{\gamma}$ is convex increasing and satisfies

$$\hat{\gamma}(\xi) > \gamma(\xi) \text{ for } \xi \in (\ell(x_m), \ell(x_M)). \tag{2.26}$$

As one has $\gamma = \hat{\gamma}$ in $[\ell(y_m), \ell(y_M)] \setminus [\ell(x_m), \ell(x_M)]$, we also have, thanks to (2.25),

$$\frac{1}{S-T}\int_E \hat{\gamma}(\ell(y(s)))\,\mathrm{d}s = \bar{u}.$$

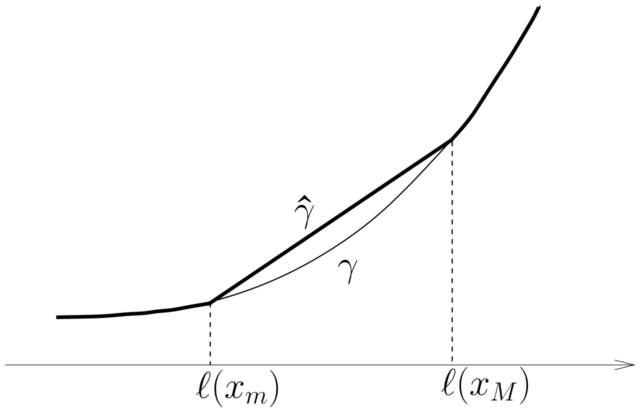


FIGURE 2.1 – *Functions $\gamma = \psi \circ \ell^{-1}$ and $\hat{\gamma}$ defined above.*

By Jensen's inequality, we obtain

$$\frac{1}{S-T}\int_E \ell(y(s))\,\mathrm{d}s \leq \hat{\gamma}^{-1}(\bar{u}). \tag{2.27}$$

Now, since $\hat{\gamma}$ is affine over $[\ell(x_m), \ell(x_M)]$, one obtains

$$\hat{\gamma}\left(\frac{1}{T}\int_0^T \ell(x(t))\,\mathrm{d}t\right) = \frac{1}{T}\int_0^T \hat{\gamma}(\ell(x(t)))\,\mathrm{d}t > \frac{1}{T}\int_0^T \gamma(\ell(x(t)))\,\mathrm{d}t = \bar{u},$$

using the fact that $x(t) \in [x_m, x_M]$ for $t \in [0,T]$, (2.26) and (2.4). Therefore, one has

$$\frac{1}{T}\int_0^T \ell(x(t))\,\mathrm{d}t > \hat{\gamma}^{-1}(\bar{u}). \tag{2.28}$$

We get by (2.24), (2.27) and (2.28)

$$\begin{aligned}
\frac{1}{S}\int_0^S \ell(y(s))\,\mathrm{d}s &= \frac{1}{S}\int_E \ell(y(s))\,\mathrm{d}s + \frac{1}{S}\int_{[0,S]\setminus E} \ell(y(s))\,\mathrm{d}s \\
&\leq \frac{S-T}{S}\hat{\gamma}^{-1}(\bar{u}) + \frac{1}{S}\int_0^T \ell(x(t))\,\mathrm{d}t \\
&< \frac{S-T}{S}\frac{1}{T}\int_0^T \ell(x(t))\,\mathrm{d}t + \frac{T}{S}\frac{1}{T}\int_0^T \ell(x(t))\,\mathrm{d}t \\
&= \frac{1}{T}\int_0^T \ell(x(t))\,\mathrm{d}t,
\end{aligned}$$

which concludes the proof. □

We now give our main result.

Theorem 3.6. *Assume that Hypotheses* (H1)-(H2)-(H3) *are fulfilled. Then, for any $T > 0$, there are two optimal solutions of* (2.13) *given by the controls $\hat{u}_T$ and*

$\check{u}_T$.

Proof. Since (H3) implies ($\bar{\text{H}}$), Proposition 3.2 gives the uniqueness of a T-admissible $B_+B_-B_+$ trajectory (see Remark 3.3), which amounts to state that there are exactly two extremals with two switches (corresponding to $n = 1$), given by the controls $\hat{u}_T(\cdot)$ and $\check{u}_T(\cdot)$. Recall that they have same cost because $\check{u}_T(\cdot)$ is obtained by a time translation of $\hat{u}_T(\cdot)$.
Now, Proposition 3.1 shows that an optimal trajectory consists in $2n$ (with $n \geq 1$) switches, that occur exactly at the maximal and minimal values. It should be noted that any such trajectory with $2n$ switches ($n \geq 1$) is $\frac{T}{n}$-periodic. By construction, an extremal has to cross $\bar{x}$ after its two first switches, say at $\bar{t} > 0$. From $t = \bar{t}$, the control alternates the same values $+1$ and -1 and switching points occur at exactly the same values of $x(\cdot)$, namely x_M and x_m. Therefore, using Cauchy-Lipschitz's Theorem, one gets $x(t) = x(t + \bar{t})$ for any $t \in [0, \bar{t}]$ and successively on the intervals $[\bar{t}, 2\bar{t}], \cdots, [(n-1)\bar{t}, n\bar{t}]$. Therefore $x(\cdot)$ is $\bar{t}$-periodic with $x(n\bar{t}) = x(T) = \bar{x}$, thus $\bar{t} = T/n$. We deduce that an extremal with $2n$ switches is T/n-periodic.
To conclude, suppose that an optimal trajectory has $2n$ switches with $n > 1$. Its cost is then equal to $J(\hat{u}_{T/n})$. Applying Lemma 3.5 with T and T/n gives $J(\hat{u}_T) < J(\hat{u}_{T/n})$, which proves that the optimal solution is achieved for $n = 1$ (*i.e*, with two switches). □

An interesting consequence of Lemma 3.5 is the monotonicity property of the cost function evaluated at the optimal solution as a function of T.

Corollary 3.1. *The optimal criterion $T \mapsto J_T(\hat{u}_T)$ is decreasing w.r.t. T.*

4 Relaxing the assumptions for local over-yielding

The previous sections have shown the crucial role played by the monotonicity property of the function ψ and the convexity of the function γ on the interval I (see Hypotheses ($\bar{\text{H}}$) and (H3)). In the present section, we consider situations for which these conditions are not fulfilled on the whole interval I but only in a neighborhood of $\bar{x}$. Typically, there could exist other values of $\bar{x}$ satisfying $\psi(\bar{x}) = \bar{u}$ (Hypothesis $(\bar{\text{H}})$ is thus not fulfilled on I) or γ could be only locally convex in a neighborhood of $\bar{x}$ (Hypothesis (H3) is thus not fulfilled on I). The idea is then to restrict the values of the period T for characterizing (periodic) optimal solutions remaining in a neighborhood of $\bar{x}$ (and presenting over-yielding). Therefore, we expect to no longer have a systematic over-yielding (see remark 4.1 and Example 5.2.2 as an illustration).

We first revisit Proposition 3.2 as follows.

Proposition 4.1. *Assume that Hypotheses* (H1)-(H2) *are fulfilled with* $\psi'(\bar{x}) > 0$. *Then there exists* $T_{max} > 0$ *such that for any* $T \in (0, T_{max})$, *there exists a unique* $(x_m, x_M) \in I^2$ *that verify* (2.20) *and* (2.21) *with*

$$(\psi(x) - \psi(\bar{x}))(x - \bar{x}) > 0, \quad \forall x \in [x_m, x_M] \setminus \{\bar{x}\}. \tag{2.29}$$

Proof. Consider a sub-interval $J := (\tilde{a}, \tilde{b}) \subset I$ with $\tilde{a} < \bar{x} < \tilde{b}$ such that the property

$$(\psi(x) - \psi(\bar{x}))(x - \bar{x}) > 0, \quad \forall x \in J \setminus \{\bar{x}\}. \tag{2.30}$$

is fulfilled (as ψ' is strictly positive at $\bar{x}$, we know that such an interval exists). Let us then consider the function $\tilde{f}$ defined on the interval $[a, b]$ by

$$\tilde{f}(x) := \left| \begin{array}{ll} f(x) & \text{if } x \in J, \\ -g(x)\left(\dfrac{\psi(\tilde{a}) + 1}{\tilde{a} - a}(x - a) - 1\right) & \text{if } x \in [a, \tilde{a}], \\ -g(x)\left(\dfrac{1 - \psi(\tilde{b})}{b - \tilde{b}}(b - x) + 1\right) & \text{if } x \in [\tilde{b}, b]. \end{array} \right.$$

Clearly, the pair $(\tilde{f}, g)$ satisfies Hypotheses (H1)-(H2)-$(\bar{\text{H}})$. The function $\tilde{f}$ is not C^1 but Lipschitz continuous, but one can easily check that Lemma 3.3, Proposition 3.2 and Lemma 3.4 are still valid with f merely Lipschitz continuous. This gives the existence and uniqueness of x_m and x_M that verify (2.20) and (2.21) for the pair $(\tilde{f}, g)$ and any $T > 0$. As $T \mapsto x_m(T)$ and $T \mapsto x_M(T)$ are respectively decreasing and increasing w.r.t. T (recall Lemma 3.4), there exists $\tilde{T} > 0$ such that $x_m(\tilde{T}) = \tilde{a}$ or $x_M(\tilde{T}) = \tilde{b}$. As f coincides with $\tilde{f}$ on $[\tilde{a}, \tilde{b}]$, we conclude that x_m, x_M are the unique numbers that verify (2.20) and (2.21) on $[\tilde{a}, \tilde{b}]$ for the pair (f, g) and any $T \leq \tilde{T}$. This can be done for any sub-interval J that verifies condition (2.30). We then consider T_{max} as the supremum of $\tilde{T}$ for all such sub-intervals J. □

Given $T < T_{max}$, one may wonder if is enough to require Hypothesis (H3) to be fulfilled on $[x_m, x_M]$ (instead of I) to obtain the optimality of the controls $\hat{u}_T$, $\check{u}_T$ as in Theorem 3.6. However, there could exist extremal trajectories taking values outside the interval $[x_m, x_M]$, without requiring additional assumption on the function ψ outside this set.
For this purpose, we consider the two controls u^- and u^+ defined by one switching time $t^- \in (0, T)$ (for u^-) and $t^+ \in (0, T)$ (for u^+) as

$$u^-(t) = \left| \begin{array}{rl} -1, & t \in [0, t^-), \\ 1, & t \in [t^-, T], \end{array} \right.$$

$$u^+(t) = \left| \begin{array}{rl} 1, & t \in [0, t^+), \\ -1, & t \in [t^+, T], \end{array} \right.$$

such that the corresponding trajectories $x(\cdot, u^-, \bar{x})$ and $x(\cdot, u^+, \bar{x})$ are T-periodic (see Fig. 2.2). Let us then define $x_T^- \in \mathbb{R}$, $x_T^+ \in \mathbb{R}$ as

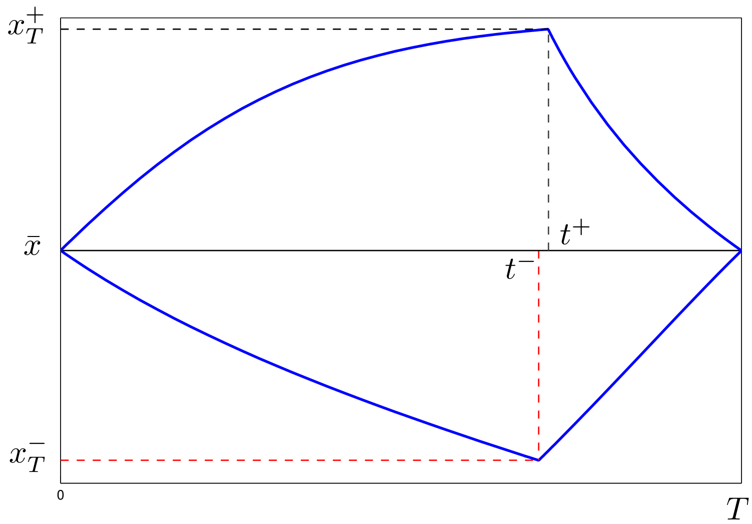


FIGURE 2.2 – *T-periodic solutions $x(\cdot, u^-, \bar{x})$ and $x(\cdot, u^+, \bar{x})$.*

$$\begin{cases} x_T^- & := x(t^+, u^+, \bar{x}), \\ x_T^+ & := x(t^-, u^+, \bar{x}). \end{cases} \tag{2.31}$$

One can check that under Hypotheses (H1)-(H2), any T-periodic solution $x(\cdot)$ of (2.1) with $x(0) = \bar{x}$ and control u taking values in $[-1, 1]$ verifies

$$x(t) \in [x_T^-, x_T^+], \quad \forall t \in [0, T]. \tag{2.32}$$

Indeed, by comparison of solutions of scalar ODEs over $[0, t^+]$, one obtains (since $u^+(t) = 1$ on $[0, t^+]$ and $f + ug \leq f + g$, $u \in [-1, 1]$):

$$x(t) \leq x(t, u^+, \bar{x}), \ \forall t \in [0, t^+].$$

Furthermore, over the time interval $[t^+, T]$, the same reasoning for the backward dynamics yields (since $u^+(t) = -1$ on $[t^+, T]$ and $-(f+ug) \leq -(f-g)$, $u \in [-1, 1]$):

$$x(t) \leq x(t, u^+, x_T^+), \ \forall t \in [t^+, T].$$

It follows that

$$x(t) \leq x(t, u^+, \bar{x}), \ \forall t \in [0, T].$$

By a similar argumentation with the control u^- in place of u^+, one concludes that

$$x(t, u^-, \bar{x}) \leq x(t) \leq x(t, u^+, \bar{x}), \ \forall t \in [0, T],$$

which completes the proof of Property (2.32). It can also be observed that one has $x_T^- < \bar{x} < x_T^+$ and $(x_T^-, x_T^+) \to (\bar{x}, \bar{x})$ when $T \to 0$.

We give now a result requiring the condition (2.29) to be fulfilled on $[x_T^-, x_T^+]$, which guarantees that any optimal solution is in the interval $[x_m, x_M]$.

Proposition 4.2. *Assume that Hypotheses* (H1)-(H2) *are fulfilled with* $\psi'(\bar{x}) > 0$. *Take* $T \in (0, T_{max})$ *such that*

$$(\psi(x) - \psi(\bar{x}))(x - \bar{x}) > 0, \quad \forall x \in [x_T^-, x_T^+] \setminus \{\bar{x}\} \tag{2.33}$$

where x_T^-, x_T^+ *are defined in* (2.31). *Then there exists unique* x_m, x_M *in* $[x_T^-, x_T^+]$ *satisfying* (2.20) *and* (2.21). *If* ψ *is increasing on* $[x_m, x_M]$, *then any* T*-admissible* *solution $x(\cdot)$ verifies*

$$\hat{x} := \max_{t\in[0,T]} x(t) \leq x_M \quad \textit{and} \quad \check{x} := \min_{t\in[0,T]} x(t) \geq x_m.$$

Proof. Fix $T \in (0, T_{max})$ that fulfills condition (2.33). Note that this is possible since ψ is increasing in a neighborhood of $\bar{x}$ and $(x_T^-, x_T^+) \to (\bar{x}, \bar{x})$ when $T \to 0$.

According to Proposition 4.1, there exists unique x_m, x_M that verify (2.20) and (2.21). Since there exists a T-admissible trajectory taking the values x_m and x_M, one has necessarily

$$x_T^- < x_m < \bar{x} < x_M < x_T^+. \tag{2.34}$$

Consider now any T-admissible solution x. From the property (2.32), one has $\hat{x} \leq x_T^+$ and $\check{x} \geq x_T^-$. Moreover, from condition (2.33) and Lemma 2.1, one has $\hat{x} > \bar{x} > \check{x}$. Let $\hat{t} \in (0, T)$ be such that $x(\hat{t}) = \hat{x}$ and suppose that one has $\hat{x} > x_M$. We can assume, without loss of generality, that $x(t) \geq \bar{x}$ is satisfied for any $t \in [0, \hat{t}]$ (if not, consider $t_0 := \sup\{t < \hat{t}\ ;\ x(t) < \bar{x}\}$ and replace $x(\cdot)$ by $x(\cdot + t_0)$). Let $(A, B) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ be defined by

$$A := \int_{\bar{x}}^{\hat{x}} \frac{\mathrm{d}x}{f(x) + g(x)} \quad \text{and} \quad B := -\int_{\bar{x}}^{\hat{x}} \frac{\mathrm{d}x}{f(x) - g(x)}.$$

It can be observed that A and B are the fastest times for a solution of (2.1) to reach, respectively, $\hat{x}$ from $\bar{x}$ (with the constant control $u = 1$) and $\bar{x}$ from $\hat{x}$ (with the constant control $u = -1$). Clearly, one has $\hat{t} \geq A$ and $T - \hat{t} > B$. We construct now a T-periodic solution $\tilde{x}(\cdot)$ of (2.1) such that $\tilde{x}(0) = \bar{x}$ and associated with a control $\tilde{u}$ defined as follows

$$\tilde{u}(t) = \left| \begin{array}{lll} \bar{u} & \text{if} & t \in [0, \hat{t} - A[, \\ 1 & \text{if} & t \in [\hat{t} - A, \hat{t}[\cup[t^\dagger, T], \\ -1 & \text{if} & t \in [\hat{t}, t^\dagger[, \end{array} \right. \tag{2.35}$$

where $t^\dagger$ is given by

$$t^\dagger = T - \int_{x^\dagger}^{\bar{x}} \frac{\mathrm{d}x}{f(x) + g(x)},$$

and $x^\dagger$ is a solution of $\kappa(x^\dagger) = T - \hat{t}$, the map $\kappa(\cdot)$ being defined by

$$\kappa(\xi) := \int_\xi^{\bar{x}} \frac{\mathrm{d}x}{f(x)+g(x)} - \int_\xi^{\hat{x}} \frac{\mathrm{d}x}{f(x)-g(x)}, \quad \xi \in I.$$

By Hypothesis (H2), the function κ is decreasing and one has

$$\kappa(x_m) = \int_{x_m}^{\hat{x}} \eta(x)\,\mathrm{d}x - A > \int_{x_m}^{x_M} \eta(x)\,\mathrm{d}x - \hat{t} = T - \hat{t},$$

and $\kappa(\bar{x}) = B < T - \hat{t}$. Therefore $x^\dagger$ is uniquely defined with $x^\dagger \in (x_m, \bar{x})$. Moreover, one has

$$t^\dagger = \hat{t} - \int_{x^\dagger}^{\hat{x}} \frac{\mathrm{d}x}{f(x)-g(x)} \in]\hat{t}, T[.$$

Expression (2.35) is thus well defined. The solution $\tilde{x}(\cdot)$ is depicted on Fig. 2.3. Clearly $\tilde{x}$ reaches $\hat{x}$ at time $\hat{t}$ and it is below the function x on the interval

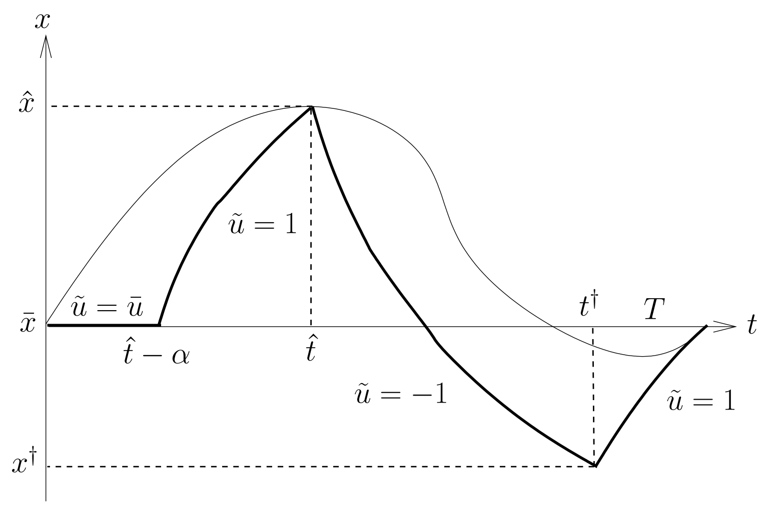


FIGURE 2.3 – *The solution $\tilde{x}$ in thick line, x in thin line.*

$[0, \hat{t}]$. On the interval $[\hat{t}, t^\dagger]$, $\tilde{x}$ has the fastest descent and therefore stays also below x on this interval. At time $t = t^\dagger$, one has $\tilde{x}(t^\dagger) = x^\dagger$. Finally, the constant control $u = 1$ is the only one that allows to connect $x^\dagger$ at time $t^\dagger$ to $\bar{x}$ at time T. So, any periodic solution has to be above $\tilde{x}$ on $[t^\dagger, T]$. We conclude that one has $x(t) \geq \tilde{x}(t)$ for any $t \in [0, T]$ (see Fig. 2.3).

As $\psi(x) > \psi(\bar{x})$ for $x \in [x_M, \hat{x}]$ and ψ is increasing on $[x_m, x_M]$, and as we have shown that $x(t) > x_m$ for any $t \in [0, T]$, one can write

$$\begin{aligned}
\int_0^T (\psi(x(t)) - \psi(\bar{x}))\,\mathrm{d}t &> \int_{\{t\in[0,T] | x(t)\leq x_M\}} (\psi(x(t)) - \psi(\bar{x}))\,\mathrm{d}t \\
&\geq \int_{\{t\in[0,T] | \tilde{x}(t)\leq x_M\}} (\psi(\tilde{x}(t)) - \psi(\bar{x}))\,\mathrm{d}t \\
&= \int_{x^\dagger}^{x_M} (\psi(x) - \psi(\bar{x}))\eta(x)\,\mathrm{d}x.
\end{aligned}$$

To conclude, since one has $x^\dagger > x_m$ and $\eta > 0$ on I, one obtains

$$\int_0^T (\psi(x(t)) - \psi(\bar{x}))\,\mathrm{d}t > \int_{x_m}^{x_M} (\psi(x) - \psi(\bar{x}))\eta(x)\,\mathrm{d}x = 0,$$

which is not possible according to Lemma 2.1. We then conclude that the inequality $\hat{x} \leq x_M$ is satisfied. In a similar manner, one can prove the other inequality $\check{x} \geq x_m$. □

For periods $T > 0$ that fulfill conditions of Proposition 4.2, we know that optimal solutions remain in the set $[x_m, x_M]$. We then obtain the same conclusion as Theorem 3.6 when Hypothesis (H3) is fulfilled on the interval $[x_m, x_M]$ only, as stated by the following Theorem.

Theorem 4.1. *Assume that Hypotheses* (H1)-(H2) *are fulfilled and consider* $T > 0$ *such that*

i) $(\psi(x) - \psi(\bar{x}))(x - \bar{x}) > 0$ *for any* $x \in [x_T^-, x_T^+] \setminus \{\bar{x}\}$*, where* x_T^-*,* x_T^+ *are defined in* (2.31)*,*

ii) ℓ *is increasing on* $[x_m, x_M]$ *and* $\gamma = \psi \circ \ell^{-1}$ *is strictly convex increasing on* $[\ell(x_m), \ell(x_M)]$*, where* x_m *and* x_M *are given by Proposition 4.1.*

Then, there are two optimal solutions of (2.13)*, given by the controls* $\hat{u}_T$ *and* $\check{u}_T$*.*

Proof. First, assumption ii) implies that ψ is increasing on the interval $[x_m, x_M]$. Thanks to i), we know from Proposition 4.2 that any extremal is such that x takes values within the interval $[x_m, x_M]$. With assumption ii) instead of Hypothesis (H3), the reader can easily check that the arguments of Theorem 3.6 apply in the same manner on $[x_m, x_M]$ (instead of the whole interval I), to prove that only the extremals $\hat{x}$ and $\check{x}$ are optimal. □

Remark 4.1. *When hypotheses (H1)-(H2)-(H3) are not all satisfied (what is considered in this section), over-yielding cannot be guaranteed for any value of T as in the previous section. However, Theorem 4.1 provides optimal periodic solutions that present over-yielding.*

5 Periodic versus constant strategies in a single population model

We consider an exploited stock of a renewable resource (fish, forest..) represented by its density $x(t)$ which follows a dynamics

$$\dot{x} = f_0(x) - E(t)x, \tag{2.36}$$

where the growth function $f_0 : \mathbb{R}_+ \to \mathbb{R}$ is of class C^1 and satisfies $f_0(0) = 0$. The harvesting effort E, which is considered as a measurable control, takes values within an interval $[0, E_{max}]$ (with $E_{max} > 0$). Such models have been extensively studied in the bio-economics literature (see for instance [Cla10] and the references cited herein). Typically an optimal steady state $\bar{x}$ associated with a constant control $\bar{E}$ is determined as maximizing a bio-economic profit of the harvesting over a discounted infinite horizon. However, it is not always possible or desirable to apply the theoretical value $\bar{E}$ of the harvesting effort in a constant manner (because of labor laws, seasonality...), but its average value is usually guaranteed on a period T. In this context, our objective is to study the impacts on the stock of applying a periodic harvesting effort instead of a constant one. We study conditions on the growth function for periodic harvesting effort having negative impact or not. In the case of negative impact, we then consider the worst scenarios to estimate the maximal loss that could be expected. There are several ways of measuring the impacts on a stock, in terms of a function $\ell(x)$ which measures the *well-being* of the stock or its *utility* (such as recreative activities). In the simplest case, $\ell(x)$ is just equal to the stock density x but more generally one can consider that $\ell : \mathbb{R}_+ \to \mathbb{R}$ is a C^1 concave increasing function.

Given a constant control $\bar{E} \in (0, E_{max})$, we then consider an associated steady-state $\bar{x}$ of (2.36) such that

$$h(\bar{x}) = \bar{E}, \tag{2.37}$$

where $h : \mathbb{R}_+ \to \mathbb{R}$ is the function defined as

$$h(x) := \left| \begin{array}{lll} \dfrac{f_0(x)}{x} & \text{if} & x > 0, \\ f_0'(x) & \text{if} & x = 0. \end{array} \right.$$

Note that equation (2.37) may have several solutions. We consider one of them which leads to a stable equilibrium (one can easily check that this amounts to have $\bar{E} > f_0'(\bar{x})$). Our aim is to study if the average criterion

$$J_T(E) := \frac{1}{T}\int_0^T \ell(x(t))\,\mathrm{d}t, \tag{2.38}$$

can be improved by considering T-periodic inputs $E(\cdot)$ satisfying

$$\frac{1}{T}\int_0^T E(t)\,\mathrm{d}t = \bar{E}, \tag{2.39}$$

and T-periodic solutions of (2.36) associated with $E(\cdot)$ with

$$x(0) = x(T) = \bar{x}. \tag{2.40}$$

In order to use the previous setting, we consider the following change of variables:

$$u := 1 - \frac{2E}{E_{max}} \ ; \ f(x) := f_0(x) - \frac{E_{max}}{2}x \ ; \ g(x) := \frac{E_{max}}{2}x,$$

and the function ψ becomes

$$\psi(x) = -\frac{f(x)}{g(x)} = 1 - \frac{2}{E_{max}}h(x).$$

So, (2.36) has exactly the form (2.1) with $u \in [-1, 1]$. Let $\bar{u} \in (-1, 1)$ be the constant control associated with $\bar{E}$ such that $\bar{u} = \psi(\bar{x})$. We now study the effects of T-periodic inputs for two growth functions f_0: the classical logistic function, and the modified one with a depensation term (that will highlight

Section 4).

5.1 The logistic growth

We recall the classical expression of this model

$$f_0(x) := rx\left(1 - \frac{x}{K}\right),$$

where $r > 0$ and $K > 0$. One can easily check that there exists a positive equilibrium $\bar{x}$ of (2.36) satisfying (2.37) as soon as $\bar{E} < r$. Moreover, $\bar{x}$ is a stable equilibrium (see [Cla10]). We assume hereafter that one has $\bar{E} < r$. Since one has

$$(f-g)(x) = x\left(r - E_{max} - \frac{r}{K}x\right) \quad ; \quad (f+g)(x) = rx\left(1 - \frac{x}{K}\right),$$

Hypotheses (H1)-(H2) are satisfied for the interval $I := (\lambda(E_{max}), K)$ where

$$\lambda(E_{max}) = \left| \begin{array}{ll} 0 & \text{if} \quad E_{max} > r, \\ h^{-1}(E_{max}) & \text{if} \quad E_{max} < r. \end{array} \right.$$

Note that ψ is an affine function: $\psi(x) = c_1 x + c_0$ with $c_0 = 1 - \frac{2r}{E_{max}}$, $c_1 = \frac{2r}{KE_{max}}$. When ℓ is strictly concave, the function $\gamma = \psi \circ \ell^{-1}$ is strictly convex (and increasing). Hypothesis (H3) is thus satisfied. According to Proposition 2.1, there is a systematic over-yielding whatever is $T > 0$, *i.e.*, the average criterion J_T is always below $\ell(\bar{x})$. Its lowest value is given by the two strategies $B_+B_-B_+$ or $B_-B_+B_-$ (see Theorem 3.6). Note that when $\ell(x) = x$, the function γ is affine and consequently the criterion J_T is always equal to $\bar{x}$, *i.e.*, the average of the stock is always equal to $\bar{x}$.

We now illustrate the over-yielding with the function

$$\ell(x) := \frac{4x}{1+x},$$

which is concave increasing. Numerical simulations have been conducted with the parameters values $r = 3$, $K = 7$, $\bar{x} = 3.5$, $E_{max} = 2.5$ and $\bar{E} = 1.5$. Results are depicted on Fig. 2.4.

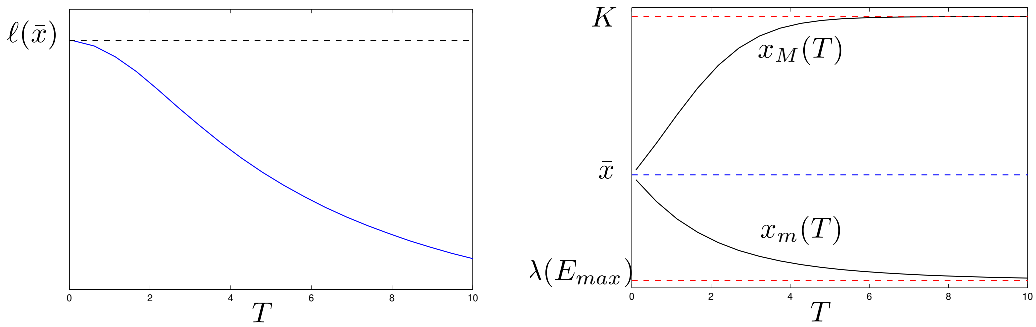


FIGURE 2.4 – *Optimal criterion $J_T(\hat{u}_T)$ (left) and x_m, x_M (right) as functions of the period T for the logistic growth.*

5.2 The logistic with depensation

Some populations are known to present a *depensation* in the first part of their growth function [Cla10], which is also called a *weak Allee effect*. This is represented by the following modification of the logistic function

$$f_0(x) := rx^\alpha \left(1 - \frac{x}{K}\right),$$

with $\alpha > 2$. For this function, one has

$$h(x) = rx^{\alpha-1} \left(1 - \frac{x}{K}\right),$$

which is increasing on $[0, x^\star)$ and decreasing on $(x^\star, K]$ with

$$x^\star := \frac{\alpha - 1}{\alpha} K,$$

(see Fig. 2.5). In presence of depensation in the model, one can also easily check that the function ψ is concave decreasing on $[0, x_c)$, convex decreasing on $(x_c, x^\star)$, and convex increasing on $(x^\star, K]$ with

$$x_c := \frac{\alpha - 2}{\alpha} K < x^\star,$$

(see Fig. 2.5).

We shall consider here the function $\ell(x) = x$ (*i.e.*, the criterion is simply

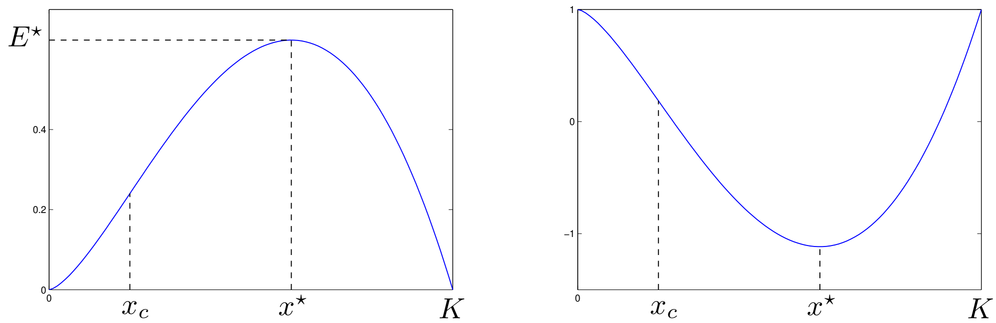


FIGURE 2.5 – *Graphs of the functions h (left) and ψ (right) for $r = 0.3$, $K = 5$, $\alpha = 2.5$, $E_{max} = 0.5893$, $E^\star = 0.6235$.*

the level of the stock x). Let us define

$$E^\star := h(x^\star).$$

We distinguish now two cases depending if E_{max} is below or above $E^\star$.

5.2.1 Case 1: $E_{max} < E^\star$

Note first that there are two solutions $\lambda_1(E_{max})$ and $\lambda_2(E_{max})$ on the interval $(0, K)$ of the equation $h(x) = E_{max}$ such that $\lambda_1(E_{max}) < x^\star < \lambda_2(E_{max})$. One can then check that Hypotheses (H1)-(H2)-(H3) are fulfilled on the interval $I := (\lambda_2(E_{max}), K)$. For any $\bar{E} \in (0, E_{max})$, one can also show, as in the logistic model, that there exists a unique solution $\bar{x} \in I$ of (2.37) which is moreover a stable steady-state of (2.36) (see [Cla10]). Proposition 2.1 guarantees then an over-yielding whatever is $T > 0$.

Fig. 2.6 depicts the optimal cost value $J_T(\hat{u}_T)$ for the following parameter values: $r = 0.3$, $K = 5$, $a = 2.5$, $\bar{x} = 4$, $\bar{E} = 0.48$, and $E_{max} = 0.5893$.

Finally, in presence of depensation in the model with a maximal harvesting effort $E_{max} < E^\star$, our analysis shows that periodic solutions cause a systematic decrease of the mean value of the stock (compared to constant harvesting).

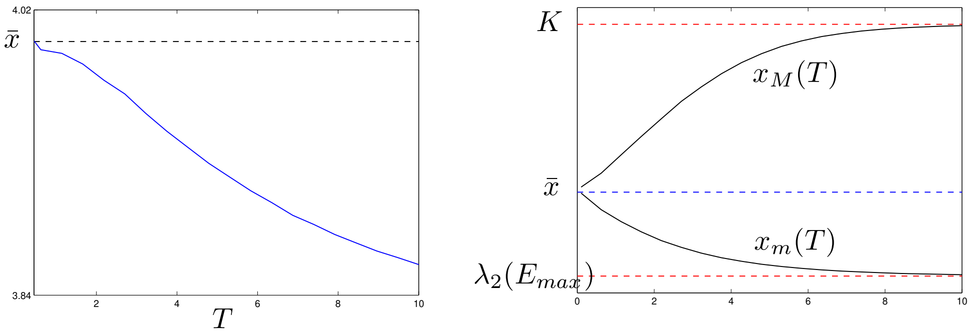


FIGURE 2.6 – *Optimal criterion $J_T(\hat{u}_T)$ (left) and x_m, x_M (right) as functions of the period T for the depensation model (case 1).*

5.2.2 Case 2: $E_{max} > E^\star$

One can easily check that Hypotheses (H1)-(H2) are fulfilled on the interval $(0, K)$, but not Hypothesis (H3). Since $\bar{x}$ is a stable steady-state of the dynamics, the point $\bar{x}$ belongs to the interval $(x^\star, K)$ (see [Cla10]). Note also that ψ is increasing in a neighborhood of $\bar{x}$. Proposition 4.1 guarantees then the existence of the T-periodic trajectory $B_+B_-B_+$ (or $B_-B_+B_-$) that satisfies the integral constraint, for T not too large. Moreover for T small enough, the function ψ is strictly convex on $[x_m(T), x_M(T)]$, and we can conclude about the optimality of these trajectories according to Theorem 4.1.

Using the same parameter values except $E_{max} = 0.8235$, the function F defined in (2.22) is depicted on Fig. 2.7 (left) for different values of T. We recall (see the proof of Proposition 3.2) that the existence of x_m, x_M is equivalent to the existence of a zero of F. One can see on this figure that T_{max} as defined in the proof of Proposition 4.1 is approximately equal to 6. For $T > 6$, we can not conclude about the existence of bang-bang trajectories, neither about their optimality. On the contrary, for $T < 6$, the $B_+B_-B_+$ and $B_-B_+B_-$ strategies are admissible and optimal, x_m, x_M, x_T^- and x_T^+ are plotted as function of T on Fig. 2.7 (right). Remark that property (2.34) is fulfilled, for all $T < 6$. Note that equation $h(x) = \bar{E}$ has two solutions $\underline{x} < \bar{x}$ (such that $\psi(\underline{x}) = \psi(\bar{x})$). Finally, on Fig. 2.8, we present the cost of the $B_+B_-B_+$ (or $B_-B_+B_-$) strategy as a function of T (for $T < 6$).

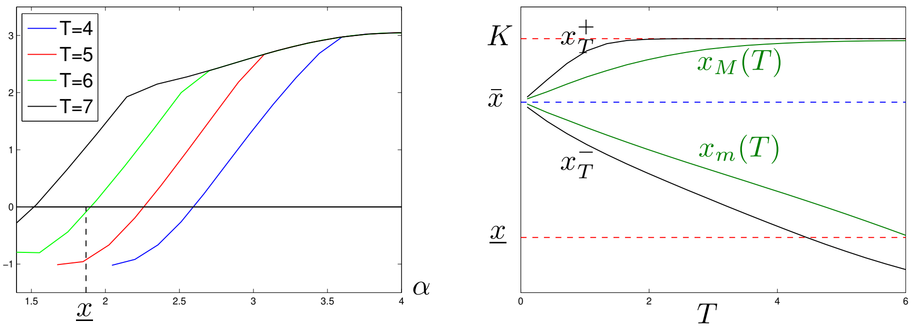


FIGURE 2.7 – *Plot of the function F defined by* (2.22) *(left), and x_m, x_M, x_T^-, x_T^+ (right) as functions of the period T $(T < 6)$ for the depensation model (case 2).*

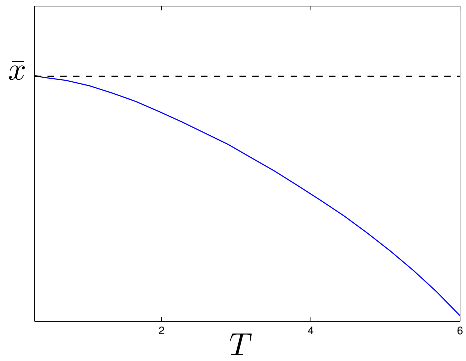


FIGURE 2.8 – *Optimal criterion $J_T(\hat{u}_T)$ for the depensation model (case 2)*

6 Conclusion

In this work, we have shown that under concavity assumptions, the optimal trajectory is the steady-state solution, that is, no over-yielding is possible.

On the contrary, under convexity assumptions, we have proved that there is exactly one optimal trajectory (up to a time translation) which is bang-bang with two switches on a period. This optimality result is global and valid for any period T. We have also relaxed the hypotheses to prove the same optimality result globally, but for a limited range of values of the period T, when only local convexity is fulfilled.

The determination of the optimal solution for large values of T when neither convexity nor concavity assumptions are fulfilled appears to be much more complex, as the bang-bang solution is no longer admissible.

This analysis was illustrated in the context of a population model subject to a harvesting effort. Depending on the growth model and the criterion, we are able to predict the effect of a periodic harvesting efforts (with the same given mean value) compared to the constant value at steady-state. Such analysis in this context is new to our best knowledge.

Some of the techniques we have proposed here to cope with the integral constraint on the control variable, which is the main characteristic of the problem we have considered, could be deployed for systems in higher dimensions, and will be the matter of a future work.

CHAPITRE 3

IMPROVEMENT OF PERFORMANCES OF CONTINUOUS BIOLOGICAL WATER TREATMENT WITH PERIODIC CONTROLS

Ce travail est soumis dans le journal *AUTOMATICA*.

Contents

1 Introduction

The continuous culture of micro-organisms is of primer importance in many industrial frameworks such as biotechnology, waste water treatment... as a way to convert raw material into products of interest or to treat pollutants in contaminated waters. During the past decades, optimization of such bio-processes has been investigated either at steady state [Bis66, WKB99], either under periodic operation [AL87, Par98]. In production optimization, one typically looks for maximizing productivity playing with the input flow rate as a control variable. It appears that periodic operations have been proved to be better than steady-state, under precise conditions. For instance, in [AL87], the π-criterion has been used to characterize the best frequency of periodic controls (improving a cost function w.r.t. its value at steady-state). For water treatment, the objectives are related to water quality and are quite different:

- either minimizing the pollutant concentration at the output of the process for a given input flow rate of water to be decontaminated (objective 1) ;
- either maximizing the input flow rate of water to be treated given a threshold of pollutant concentration not be exceeded at the output (objective 2).

Classically, such processes are well represented with the chemostat model [SW95, HLRS17]:

$$\left\{ \begin{array}{rcl} \dot{s} & = & -\dfrac{1}{Y}\mu(s,x)x + D(s_{in}-s), \\ \dot{x} & = & \mu(s,x)x - Dx, \end{array} \right. \tag{3.1}$$

where s and x denote respectively the substrate (here the pollutant) and biomass concentrations in a tank of constant volume V. The dilution rate $D = F/V$ (where F is the input flow rate of the contaminated water) is the control variable, Y the conversion rate, s_{in} the input substrate concentration and $\mu(\cdot,\cdot)$ the specific growth rate of micro-organisms. For sake of generality, we leave open the possibility for the growth function to be density-dependent or not, i.e. μ depends on s only or also on x. Note that equilibria

$(\bar{s}, \bar{x})$ of (3.1) satisfy $\bar{x} + Y\bar{s} = s_{in}$ and $\mu(\bar{s}, Y(s_{in} - \bar{s})) = \bar{D}$, that uniquely link the flow rate $\bar{D}$ to the output concentration $\bar{s}$ (when the latter equation admits more than one solution, one considers only the smallest positive one, which necessarily corresponds to a stable equilibrium, see e.g. [HLRS17]). So, given a dilution rate $\bar{D}$ in objective 1, or given a threshold $\bar{s}$ in objective 2, there is no possible improvement. However, if one considers periodic solutions of (3.1) with periodic controls $D(\cdot)$ over a given time period T, the total quantity of water treated during a period is $Q_T := \langle D \rangle_T T$ ($\bar{Q} := \bar{D}T$ for steady-states), where

$$\langle \xi \rangle_T := \frac{1}{T} \int_t^{t+T} \xi(\tau) \, \mathrm{d}\tau,$$

denotes the average of any T-periodic function $\xi(\cdot) \in L^1_{loc}$. One then looks for the two control problems:

Problem 1. *Given a quantity of water $\bar{Q}$ to be treated during a period T, does there exist a non-constant T-periodic solution such that $\langle s \rangle_T \leq \bar{s}$ and $\langle D \rangle_T \geq \bar{D}$?*

In connection with Prob. 1, we shall also investigate the optimal control problem

$$\inf_{D(\cdot)} \langle s \rangle_T \quad \text{s.t. } s(0) = s(T) \text{ and } \langle D \rangle_T \geq \bar{D}, \tag{3.2}$$

where $D(\cdot)$ is a measurable control taking values in $[D_-, D_+]$ with $0 \leq D_- < \bar{D} < D_+$, and $s(\cdot)$ satisfies (1.4).

Problem 2. *Given a threshold $\bar{s}$, does there exist a non-constant T-periodic solution such that $\langle D \rangle_T \geq \bar{D}$ and $\langle s \rangle_T \leq \bar{s}$?*

Similarly, we shall consider the optimal control problem

$$\sup_{D(\cdot)} \langle D \rangle_T \quad \text{s.t. } s(0) = s(T) \text{ and } \langle s \rangle_T \leq \bar{s}. \tag{3.3}$$

Typically, in waste water treatment industry, the quality of the treated water is not always measured instantaneously but averaged over a time period depending on the final destination of the treated water (housing, industry, agriculture...). In Pb. 1, the control $D(\cdot)$ satisfies an integral constraint, while in Pb. 2 there is an integral constraint over $s(\cdot)$. Hence, Pb. 2 can be seen as a kind of "dual" of Pb. 1. This is quite different from what is studied in the biochemical literature when optimizing the productivity without integral constraint on the control, as recalled previously. To our knowledge, such problems have not been yet studied in the literature. Preliminary results on these questions have been given in the conference paper [BRT18], where Pb. 1 only is considered for a single class of growth functions, and solutions of (3.2) were conjectured.

The paper is organized as follows. In Section 2, we introduce our main assumptions on the kinetics. In Section 3, we discuss the existence of solutions of Prob. 1-2. When improvement with periodic solutions is possible, we aim at quantifying the maximal improvement as a function of the period in Section 4. Doing so, we apply recent results about optimal control for scalar dynamics under integral constraint on the input [BRTss] for Prob. 1, and give an extension of these results for Prob. 2. Numerical simulations illustrate the possible gains in Section 5. Finally, some results of [BRTss] are recalled in the Appendix.

2 Main assumptions

Since we only only deal with periodic solutions of (3.1), we consider in the sequel the simplified dynamics for the variable $s(\cdot)$ only:

$$\dot{s} = (-\nu(s) + D(t))(s_{in} - s), \tag{3.4}$$

where $\nu(s) := \mu(s, s_{in} - s)$, assuming without any loss of generality $Y = 1$, and recall that $D(\cdot)$ is a measurable control with values in $[D_-, D_+]$. We shall consider the dynamics (3.4) on $(0, s_{in})$ and $\bar{D} \in (D_-, D_+)$ is chosen in such a way that

$$\bar{s} := \inf\{s \ ; \ \nu(s) > \bar{D}\} < s_{in}. \tag{3.5}$$

This choice of $\bar{D}$ avoids washout of biomass (i.e. $x = 0$ at steady-state). Let us now introduce the following (minimal) assumption on ν.

Hypothesis 2.1. *The function ν is Lipschitz, non-negative, and null only at 0. The number of solutions to the equation $\nu(s) = \bar{D}$ over $(0, s_{in})$ (with $\bar{D} \in (D_-, D_+)$)*

is finite.

Remark 2.1. *Under Hyp. 2.1 and* (3.5)*, ν is increasing in a neighborhood of $\bar{s}$ which is then a locally stable equilibrium of* (3.4) *for the constant control $D = \bar{D}$.*

To cover a large variety of growth functions, we introduce three kinds of hypotheses.

Hypothesis 2.2. *The function ν is strictly convex in a neighborhood of $s = \bar{s}$.*

Hypothesis 2.3. *The function ν is strictly convex over $(0, s_{in})$.*

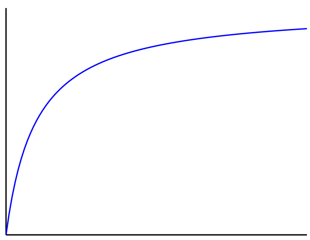
Hypothesis 2.4. *There is $\bar{\nu} \in C^0([0, s_{in}] \ ; \ \mathbb{R})$ such that*

- *$\bar{\nu} \geq \nu$ over $(0, s_{in})$ with $\bar{\nu}(\bar{s}) = \nu(\bar{s})$,*
- *$\bar{\nu}$ is concave non decreasing over $(0, s_{in})$.*

Such hypotheses are satisfied by the following kinetics (commonly found in the literature):

- Monod's kinetics [Mon50], as an increasing function of s:

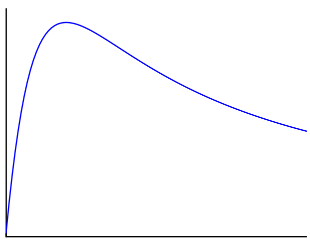
$$\mu(s) = \frac{\mu_{max} s}{K_s + s}.$$



Plot of μ.

- Haldane's kinetics [And68], which models an inhibition for large value of s:

$$\mu(s) = \frac{\mu_m s}{K_s + s + s^2/K_I}.$$

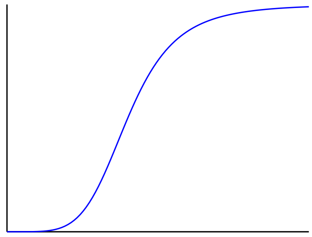


Plot of μ.

Its maximum is reached at $\hat{s} := \sqrt{K_s K_i})$.

- Hill's kinetics [Mos58], which exhibits a weak Allee effect for small value of s:

$$\mu(s) = \frac{\mu_{max} s^n}{K_s^n + s^n}, \quad (n \in \mathbb{N}^*)$$

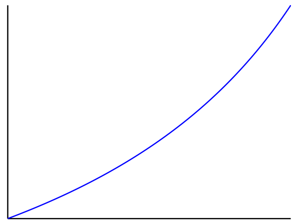


Plot of μ.

  and μ changes it concavity at $s_c := K_s \left(\frac{n-1}{n+1}\right)^{1/n}$.

- Contois's kinetics [Con59], which is density dependent:

$$\mu(s,x) = \frac{\mu_{max} s}{Kx + s},$$



Plot of ν ($K < 1$).

  and ν is strictly convex for $K < 1$, concave for $K \geq 1$.

We shall next see that, depending on the kinetics, improvement of the criteria is possible or not.

3 Conditions for improvements

Our first objective is to study the existence of non-constant solutions of Prob. 1-2.

Lemma 3.1. *Given a pair $(\bar{s}, \bar{D})$ satisfying (3.5), for any $T > 0$ there are $\varepsilon_0 > 0$ and $C > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, there exists a non-constant periodic solution $s(\cdot)$ of (3.4) such that $\langle D \rangle_T = \bar{D}$ or $\langle s \rangle_T = \bar{s}$ with $\|s - \bar{s}\|_\infty \leq C\varepsilon$.*

Proof. We show first the existence of non-constant solutions of Prob. 1. Let $v(\cdot)$ be a T-periodic measurable bounded function with $\langle v \rangle_T = 0$, non null

almost everywhere. Consider the control $D_\varepsilon(\cdot) := \bar{D} + \varepsilon v(\cdot)$, which takes values in $[D_-, D_+]$ for $\varepsilon > 0$ small enough (say $0 < \varepsilon \le \varepsilon_1$), and verifies $\langle D_\varepsilon \rangle_T = \bar{D}$. Let $\theta(s_0, \varepsilon) := s(T, D_\varepsilon, s_0) - s_0$, where $s(t, D, s_0)$ denotes the solution of (3.4) at time t with $s(0) = s_0$ and control $D(\cdot)$. By continuous dependency of $s(T, D_\varepsilon, s_0)$ w.r.t. (s_0, ε), θ is continuous. From Rem. 2.1, ν has to be increasing in any sufficiently small neighborhood (s_0^-, s_0^+) of $\bar{s}$, which implies that $\theta(s_0^-, 0) > 0$, $\theta(s_0^+, 0) < 0$ and thus $\theta(s_0^-, \varepsilon) > 0$, $\theta(s_0^+, \varepsilon) < 0$ for ε sufficiently small (say $0 < \varepsilon \le \varepsilon_2$) By the Mean value Theorem, we deduce the existence of $\tilde{s}_0 \in (s_0^-, s_0^+)$ such that $\theta(\tilde{s}_0, \varepsilon) = 0$, that is, the existence of a non-constant T-periodic solution $\tilde{s} := s(\cdot, D_\varepsilon, \tilde{s}_0)$ with $\langle D_\varepsilon \rangle_T = \bar{D}$. Finally, notice that one has $s(\cdot, \bar{D}, \bar{s}) = \bar{s}$ and thus, Gronwall's Lemma implies the existence of a constant $C_1 > 0$ (depending on T and v) such that $||\tilde{s} - \bar{s}||_\infty \le C_1 \varepsilon$ for any $\varepsilon \in (0, \varepsilon_0']$ with $\varepsilon_0' := \min(\varepsilon_1, \varepsilon_2)$.
We now turn to (3.3). Let $y \in C^1(\mathbb{R}, \mathbb{R})$ be a T-periodic function such that $\langle y \rangle_T = 0$ (and non identically null). For ε small enough (say $0 < \varepsilon \le \varepsilon_3$), $t \mapsto s_\varepsilon(t) := \bar{s} + \varepsilon y(t)$ is with values in $(0, s_{in})$ and satisfies $||s_\varepsilon - \bar{s}||_\infty < \|y\|_\infty \varepsilon$ as well as $\langle s_\varepsilon \rangle_T = \bar{s}$. Notice that $s_\varepsilon(\cdot)$ is a solution of (3.4) for the control $D_\varepsilon(t) := \frac{\dot{s}_\varepsilon(t)}{s_{in} - s_\varepsilon(t)} + \nu(s_\varepsilon(t))$. One then has $|D_\varepsilon(t) - \bar{D}| \le F(t, \varepsilon) := \varepsilon \left| \frac{\dot{y}(t)}{s_{in} - \bar{s} - \varepsilon y(t)} \right| + \varepsilon L |y(t)|$, where L is the Lipschitz constant of ν. As F tends to 0 when ε tends to 0, uniformly in t, we conclude that D_ε is admissible for ε small enough (say $0 < \varepsilon \le \varepsilon_4$), and thus s_ε is a non-constant periodic solution with $\langle s_\varepsilon \rangle_T = \bar{s}$ and $||s_\varepsilon - \bar{s}||_\infty < \|y\|_\infty \varepsilon$ for $0 < \varepsilon < \varepsilon_0'' := \min(\varepsilon_3, \varepsilon_4)$.
This concludes the proof of the lemma taking $C := \max(C_1, \|y\|_\infty)$ and $\varepsilon_0 := \min(\varepsilon_0', \varepsilon_0'')$. □

Remark 3.1. *Since the optimal control problems* (3.2) *and* (3.3) *involve inequality constraints, this lemma shows the existence of admissible solutions for these problems.*

In the sequel, "equality constraint" in Prob. 1, resp. Prob. 2 means that one considers solutions with $\langle D \rangle_T = \bar{D}$, resp. $\langle s \rangle_T = \bar{s}$.

Proposition 3.1. *If* (3.5) *and Hyp. 2.1 are verified, then:*

(i) *if Hyp. 2.2 is fulfilled then for any $T > 0$, Prob. 1 and 2 admit solutions;*

(ii) *if Hyp. 2.3 is fulfilled then for any $T > 0$, any non-constant periodic solution with equality constraint gives an improvement for both problems;*

(iii) *if Hyp. 2.4 is satisfied, there is no solution to Prob. 1 and 2.*

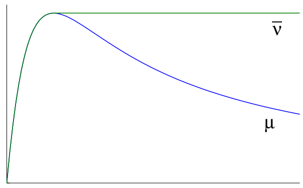
Moreover any periodic solution verifies $\langle D\rangle_T = \langle \nu(s)\rangle_T$.

Proof. Given a non-constant T-periodic solution $s(\cdot)$ of (3.4) over $(0, s_{in})$, $t \mapsto \ln(s_{in} - s(t))$ is also periodic. From (3.4) one obtains $\langle D\rangle_T = \langle \nu(s)\rangle_T$. Recall now from Remark 2.1 that ν is increasing in a neighborhood of $\bar{s}$.
Proof of (i). Under Hyp. 2.2, the periodic solution $s(\cdot)$ can be chosen in such a way that ν is strictly convex increasing over $s([0, T])$ (Lemma 3.1). Jensen's inequality then gives $\langle \nu(s)\rangle_T > \nu(\langle s\rangle_T)$. In Prob. 1, one has $\nu(\bar{s}) = \bar{D} = \langle D\rangle_T > \nu(\langle s\rangle_T)$ and since ν is increasing over $s([0, T])$, we deduce that the inequality $\bar{s} > \langle s\rangle_T$ is verified. In Prob. 2, one has $Q_T/T = \langle D\rangle_T > \nu(\langle s\rangle_T) = \nu(\bar{s}) = \bar{Q}/T$ and thus the inequality $Q_T > \bar{Q}$ is fulfilled.
Proof of (ii). Under Hyp. 2.3, Hyp. 2.1 implies that ν is increasing over $(0, s_{in})$ and the former inequalities are then satisfied for any non-constant periodic solution with values in $(0, s_{in})$.
Proof of (iii). Under Hyp. 2.4, Jensen's inequality applied to the concave function $\bar{\nu}$ implies that $\langle \bar{\nu}(s)\rangle_T \leq \bar{\nu}(\langle s\rangle_T)$, and since $\bar{\nu} \geq \nu$, one obtains $\langle \nu(s)\rangle_T \leq \bar{\nu}(\langle s\rangle_T)$. In Prob. 1, one has $\bar{\nu}(\bar{s}) = \nu(\bar{s}) = \bar{D} = \langle D\rangle_T = \langle \nu(s)\rangle_T$. One then obtains $\bar{\nu}(\bar{s}) \leq \bar{\nu}(\langle s\rangle_T)$ from which we deduce the inequality $\bar{s} \leq \langle s\rangle_T$, since $\bar{\nu}$ is non decreasing. In Prob. 2, one has $Q_T/T = \langle D\rangle_T = \langle \nu(s)\rangle_T$ and $\bar{\nu}(\langle s\rangle_T) = \bar{\nu}(\bar{s}) = \nu(\bar{s}) = \bar{Q}/T$. One then obtains $Q_T \leq \bar{Q}$ because $\langle D\rangle_T \geq \bar{D}$. In any case, no improvement is possible. □

Let us now come back to the four growth functions listed above to examine how to apply Proposition 3.1.

- Monod's function is concave increasing and Hyp. 2.4 is fulfilled with $\bar{\nu} = \mu$. Thus, no improvement is possible.
- Haldane's function is neither convex neither concave. However, from (3.5) one has $\bar{s} \in (0, \hat{s})$ and μ is concave increasing over $[0, \hat{s}]$. Then, the function

$$\bar{\nu}(s) := \begin{cases} \mu(s), & s < \hat{s} \\ \mu(\hat{s}), & s \geq \hat{s} \end{cases}$$

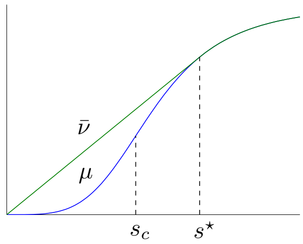


graphs of μ and $\bar{\nu}$

fulfills Hyp. 2.4, so, no improvement is possible.

- Hill's function also changed of concavity but is increasing. Its concave envelope on $\mathbb{R}_+$ is given by the function

$$\bar{\nu}(s) := \begin{cases} \mu'(s^*)s, & s < s^* \\ \mu(s), & s \geq s^* \end{cases}$$



graphs of μ and $\bar{\nu}$

where s^* is the unique abscissa whose tangent to the graph of μ passes through the origin: $s^* := s_c(1+n)^{\frac{1}{n}}$. If $\bar{s} < s_c$, μ is locally convex and Hyp. 2.2 is satisfied: improvement is possible with periodic solutions that belongs to $(0, s_c)$. If $\bar{s} \geq s^*$, $\bar{\nu}$ satisfies Hyp. 2.4 and no improvement is possible.

- For the Contois function, Hyp. 2.3 is fulfilled for $K < 1$: any non-constant periodic solution improves both criteria. For $K \geq 1$, Hyp. 2.4 is satisfied with $\bar{\nu} = \nu$: no improvement is possible.

4 Optimal improvements

For a given period T, we now wish to characterize periodic solutions belonging to the domain where ν is increasing and strictly convex that provide the best improvement in Prob. 1 and 2 (typically for μ of Contois or Hill type). Doing so, we are given a pair $(\bar{s}, \bar{D})$ satisfying (3.5), and we assume throughout this section that either Hyp. 2.2 or Hyp. 2.3 is satisfied.

Lemma 4.1. *For any periodic solution $s(\cdot)$ of* (3.4) *such that $\langle s \rangle_T < \bar{s}$ and $\langle D \rangle_T \geq \bar{D}$, there exists $t \in [0, T]$ with $s(t) = \bar{s}$.*

Proof. Using (3.4) and $\nu(\bar{s}) = \bar{D}$, we get

$$\int_0^T [\nu(s(t)) - \nu(\bar{s})]\,\mathrm{d}t \geq 0. \tag{3.6}$$

Then, supposing that $s(t) < \bar{s}$ for any time $t \in [0, T]$ gives a contradiction with (3.6) using that ν is increasing over $s([0, T])$. □

We can then consider without loss of generality solutions of (3.4) with $s(0) = \bar{s}$. Related to Prob. 1, we define the "value function", for $\bar{D} \in [D_-, D_+]$:

$$V_T(\bar{D}) := \inf_{D(\cdot)} \left\{ \langle s \rangle_T \,;\, s(0) = s(T) = \bar{s},\ \langle D \rangle_T = \bar{D} \right\},$$

where $s(\cdot)$ is a solution of (3.4) associated with $D(\cdot)$. The usual argumentation on compactness of the set of admissible solutions and continuity of the cost function guarantees the existence of an optimal control (and thus that the infimum can be replaced by a minimum in the previous display), see, e.g., [BP07].

Proposition 4.1. *The mapping V_T is the value function of* (3.2) *and it is increasing.*

Proof. Suppose that an optimal control $D(\cdot)$ (non-constant) satisfies $\langle D \rangle_T > \bar{D}$. Denote by $s(\cdot)$ its associated T-periodic solution and let $E := \{t \in [0, T] \,;\, D(t) > \bar{D}\}$ which is necessarily such that $\text{meas}(E) > 0$. Set

$$\gamma := \min \left(\frac{\langle D \rangle_T - \bar{D}}{\text{meas}(E)}, \bar{D} - D_- \right) > 0,$$

and define a control $\tilde{D}$ on $[0, T]$ as

$$\tilde{D}(t) := \begin{cases} D(t) - \gamma & \text{if} \quad t \in E, \\ D(t) & \text{if} \quad t \notin E. \end{cases}$$

The control $\tilde{D}$ is with values in $[D_-, D_+]$ and so, it is admissible. In addition, one has $\bar{D} \leq \langle \tilde{D} \rangle_T < \langle D \rangle_T$. Let $\tilde{s}(\cdot, s_0)$ be the unique solution of (3.4) associated with $\tilde{D}(\cdot)$ and such that $\tilde{s}(0, s_0) = s_0$. One has $\tilde{s}(T, 0) > 0$ because $\tilde{s}(\cdot, 0)$ is non-negative and $\langle \tilde{D} \rangle_T > 0$ implies that $\tilde{s}(\cdot, 0)$ cannot be

identically null on $[0, T]$. Moreover, one has $\tilde{D}(t) \leq D(t)$ for $t \geq 0$ and since $\text{meas}\{t \in [0, T];\, \tilde{D}(t) < D(t)\} = \text{meas}(E) > 0$, we deduce that $\tilde{s}(T, \bar{s}) < \bar{s}$ (by comparison of solutions of scalar differential equations, see e.g. [Wal98]). Thanks to the Mean Value Theorem applied to the continuous function $s_0 \mapsto \tilde{s}(T, s_0) - s_0$, one deduces the existence of $\tilde{s}_0 \in (0, \bar{s})$ such that $\tilde{s}(T, \tilde{s}_0) = \tilde{s}_0$. The solution $\tilde{s}(\cdot, \tilde{s}_0)$ (associated to $\tilde{D}$) is T-periodic and verifies $\tilde{s}(t, \tilde{s}_0) < s(t)$ for any $t \in [0, T]$ (by comparison of solutions). Therefore, one gets $\langle \tilde{s}(\cdot, \tilde{s}_0)\rangle_T < \langle s \rangle_T$ and we can conclude that D is not optimal for (3.2) with $\langle D \rangle_T > \bar{D}$ (since $\langle \tilde{D} \rangle_T < \langle D \rangle_T$). Hence, the inequality constraint in (3.2) must be saturated.
Let now $\bar{D}_1, \bar{D}_2 \in [D_-, D_+]$ be such that $0 < \bar{D}_1 < \bar{D}_2$. Since an optimal solution of (3.2) necessarily saturates the inequality constraint, $V_T(\bar{D}_1)$ is the value function associated with (3.2) which means that

$$V_T(\bar{D}_1) := \min_{D(\cdot)} \left\{ \langle s \rangle_T \,;\, s(0) = s(T) = \bar{s},\ \langle D \rangle_T \geq \bar{D}_1 \right\}.$$

If follows that an optimal pair $(D_2(\cdot), s_2(\cdot))$ for $V_T(\bar{D}_2)$ is admissible for $V_T(\bar{D}_1)$ (since $\langle D_2 \rangle_T = \bar{D}_2 \geq \bar{D}_1$) which implies that $\langle s_2 \rangle_T \geq V_T(\bar{D}_1)$ and thus the result. □

Remark 4.1. *For Prob. 1, if $T > 0$ and $\bar{D} \in (D_-, D_+)$ are such that an improvement exists, then one has $V_T(\bar{D}) < \bar{s}$ where $\bar{s}$ is defined in* (3.5).

Let I be a sub-interval of $[0, s_{in}]$ containing $\bar{s}$ defined by (3.5), that is invariant by (3.4) for any control $D(\cdot) \in [D_-, D_+]$. We consider bang-bang controls:

$$\hat{D}_T(t) := \begin{cases} D_+, & 0 \leq t < t_1, \\ D_-, & t_1 \leq t < t_2, \\ D_+, & t_2 \leq t < T, \end{cases} \tag{3.7}$$

(where $0 < t_1 < t_2 < T$) and posit $s_M = s(t_1)$, $s_m = s(t_2)$, which belong to I. One can check that a solution $s(\cdot)$ of (3.4) with $s(0) = \bar{s}$ and control $\hat{D}_T(\cdot)$ is T-periodic if and only if one has

$$\int_{s_m}^{s_M} \eta(s)\, \mathrm{d}s = T, \tag{3.8}$$

where the function $\eta : I \to \mathbb{R}$ is defined as

$$\eta(s) := \frac{1}{(D_+ - \nu(s))(s_{in} - s)} - \frac{1}{(D_- - \nu(s))(s_{in} - s)}.$$

In the same way, $\hat{D}(\cdot)$ satisfies $\langle \hat{D} \rangle_T = \bar{D}$ if and only if

$$\int_{s_m}^{s_M} \eta(s)\nu(s)\,\mathrm{d}s = \bar{D}T. \tag{3.9}$$

To conclude about the optimality of a control of type (3.7), we shall apply recent results [BRTss] on periodic optimal control problems governed by a scalar dynamics under an L^1-norm constraint on the control (see Appendix 7).

Proposition 4.2. *For any $\bar{D} \in (D_-, D_+)$, there exists a unique pair $(s_m, s_M) \in (0, s_{in})^2$ satisfying* (3.8)-(3.9) *and $\hat{D}_T(\cdot)$ with $t_1 := \inf\{t > 0,\ s(t) = s_M\}$, $t_2 := \inf\{t > t_1,\ s(t) = s_m\}$ is an optimal control for Prob. 1*

(i) *for any $T > 0$ if ν is convex increasing on I,*

(ii) *for $T > 0$ not too large if ν is locally convex increasing near $\bar{s}$.*

Proof. Posit $u := aD + b \in [-1, 1]$ with

$$a := \frac{2}{D_+ - D_-},\quad b := -\frac{D_+ + D_-}{D_+ - D_-}$$

and define for $s \in I$:

$$\begin{aligned} f(s) &:= (-\nu(s) - b/a)(s_{in} - s), \\ g(s) &:= (s_{in} - s)/a, \\ \ell(s) &:= s, \\ \psi(s) &:= a\nu(s) - b, \end{aligned}$$

so that (3.4) rewrites $\dot{s} = f(s) + ug(s)$. One can also check that we are exactly in the conditions of Theorem A1 and A3 given in Appendix 7. Therefore $\hat{D}_T(\cdot)$ is optimal. □

We now turn to Prob. 2 which involves an integral constraint on the state and not on the control. Therefore, the results recalled in Appendix 7 do not apply. Indeed, an integral constraint on the state is more difficult to grasp than for the control, as it cannot be formulated regardless of the dynamics, and extending the results in [BRTss] is not straightforward. Nevertheless, we show that there exists a form of *duality* between Prob. 1 and 2. Doing so, we define the "dual" value function, for $\bar{s} \in I$:

$$W_T(\bar{s}) := \sup_{D(\cdot)} \left\{ \langle D \rangle_T \, ; \, s(0) = s(T), \; \langle s \rangle_T = \bar{s} \right\}.$$

Similarly to V_T, the above supremum can be replaced by a maximum (see [BP07]).

Proposition 4.3. *For any $\bar{s} \in I$ and T that satisfy conditions of Proposition 4.2, one has*

$$W_T(\bar{s}) = \max\{\bar{D} \in [D_-, D_+]; \; V_T(\bar{D}) = \bar{s}\} = V_T^{-1}(\bar{s}), \qquad (3.10)$$

and W_T is the value function of (3.3).

Proof. Assume first that ν is convex increasing on I and let us show that V_T is continuous on (D_-, D_+) for any $T > 0$. For any $\bar{D} \in (D_-, D_+)$, there exists a unique pair $(s_m, s_M) \in (0, s_{in})^2$ satisfying (3.8)-(3.9), that is,

$$F(s_m, s_M, \bar{D}) := \begin{bmatrix} \int_{s_m}^{s_M} \eta(s)ds - T \\ \int_{s_m}^{s_M} \eta(s)\nu(s)ds - \bar{D}T \end{bmatrix} = 0,$$

and the Jacobian matrix of F w.r.t. (s_m, s_M)

$$\begin{bmatrix} -\eta(s_m) & \eta(s_M) \\ -\eta(s_m)\nu(s_m) & \eta(s_M)\nu(s_M) \end{bmatrix}$$

is non singular. By the Implicit Function Theorem, s_m and s_M are C^1 functions of $\bar{D}$, and $\hat{D}_T(\cdot)$ is then continuous in L^1 w.r.t. $\bar{D}$. Recall that the map $D(\cdot) \mapsto s(\cdot, D)$ is continuous from L^1 into C^0 (see e.g. Theorem 4.2 in [BP07]), so that $\bar{D} \mapsto s(\cdot, \hat{D}_T)$ is continuous, and thus V_T as well. Write $I = [s_-, s_+]$. As I is invariant and ν increasing, one has necessarily $V_T(D_-) = s_-$, $V_T(D_+) = s_+$. Since V_T is continuous and increasing (Proposition 4.1), it is thus invertible on I with $V_T^{-1}(I) = [D_-, D_+]$. Take $\bar{s} \in I$ and let $D^\dagger := V_T^{-1}(\bar{s})$. Let $D(\cdot)$ be an optimal control for Prob. 1 (i.e. such that $\langle D \rangle_T = D^\dagger$), which generates a solution $s(\cdot)$ with $s(T) = s(0)$ and $\langle s \rangle_T = \bar{s}$. The control $D(\cdot)$ is then

sub-optimal for Prob. 2, i.e. $W_T(\bar{s}) \geq \langle D \rangle_T = D^\dagger$. Suppose now that there exists an optimal control $\tilde{D}(\cdot)$ for Prob. 2 with $\langle \tilde{D} \rangle_T > D^\dagger$, and let $\tilde{s}(\cdot)$ be the associated solution satisfying the constraint $\langle \tilde{s} \rangle_T \leq \bar{s}$. Since V_T is increasing, one gets $V_T(\langle \tilde{D} \rangle_T) > V_T(D^\dagger)$. However, by definition of V_T, one has $V_T(\langle \tilde{D} \rangle_T) \leq \langle \tilde{s} \rangle_T \leq \bar{s}$, leading to a contradiction. We conclude that one has necessarily $W_T(\bar{s}) = D^\dagger$. As V_T is increasing, an optimal solution for (3.3) has to saturate the constraint $\langle s \rangle_T \leq \bar{s}$, and thus W_T is value function.
If ν is convex increasing only over a sub-interval $J \subset I$ with $\bar{s} \in J$, consider any increasing convex function $\bar{\nu}$ which coincides with ν on J. Denote by $\bar{V}_T$, $\bar{W}_T$ the corresponding functions. One then has

$$\bar{W}_T(\bar{s}) = \max\{\bar{D}\,;\, \bar{V}_T(\bar{D}) = \bar{s}\} = \bar{V}_T^{-1}(\bar{s}).$$

For T small enough, V_T and $\bar{V}_T$ coincide in a neighborhood of $\bar{s}$, and W_T, $\bar{W}_T$ as well in a neighborhood of $\bar{D} = \nu(\bar{s})$. Thus, $V_T^{-1}(\bar{s})$ is non empty and as V_T is increasing, $V_T^{-1}(\bar{s})$ is unique, equal to $\bar{V}_T^{-1}(\bar{s})$ and one has also $\max\{\bar{D}\,;\, V_T(\bar{D}) = \bar{s}\} = V_T^{-1}(\bar{s})$. Finally, (3.10) is fulfilled for T not too large. □

For the bang-bang controls (3.7), the constraint $\langle s \rangle_T = \bar{s}$ can be written, similarly as done before, with

$$\int_{s_m}^{s_M} s\,\eta(s)\,\mathrm{d}s = \bar{s}\,T. \tag{3.11}$$

Proposition 4.2 and 4.3 lead then to the following characterization of optimal solutions for Prob. 2.

Proposition 4.4. *There exists a unique pair* $(s_m, s_M) \in (0, s_{in})^2$ *satisfying* (3.8)-(3.11), *and* $\hat{D}_T(\cdot)$ *with* $t_1 := \inf\{t > 0,\ s(t) = s_M\}$, $t_2 := \inf\{t > t_1,\ s(t) = s_m\}$ *is an optimal control for Prob. 2*

(i) *for any* $T > 0$ *if* ν *is convex increasing on* I,

(ii) *for* $T > 0$ *not too large, if* ν *is locally convex increasing near* $\bar{s}$.

Remark 4.2. *If an improvement exists for some* $T > 0$ *and* $\bar{s} \in I$ *in Prob. 2, then one has* $W_T(\bar{s}) > \bar{D}$ *where* $\bar{D}$ *is defined in* (3.5).

5 Numerical illustrations

Let us first examine how the optimal switching times t_1, t_2 of $\hat{D}_T$ can be computed. For Prob. 1, the constraint $\langle D\rangle_T = \bar{D} = \nu(\bar{s})$ applied to the bang-bang control (3.7) imposes a relation between t_1, t_2 that we write as follows.

$$t_2 = t_2(t_1) := t_1 + T\frac{D_+ - \bar{D}}{D_+ - D_-}.$$

From Proposition 4.2, t_1 is unique and can be then determined as the unique zero of the map $t_1 \mapsto s_{t_1,t_2(t_1)}(T) - \bar{s}$ on $(0,T)$, where $s_{t_1,t_2}(\cdot)$ is the unique solution of (3.4) associated with $\hat{D}_T$ such that $s(0) = \bar{s}$. We present below numerical simulations of optimal solutions of Prob. 1 and Prob. 2 for Contois's kinetics, and then for Hill's kinetics.

For Contois's kinetics with $K > 1$ (which is convex increasing), for any $\bar{D} \in (D_-, D_+)$ and $T > 0$ one has $V_T(\bar{D}) < \bar{s} = \nu^{-1}(\bar{D})$ since the improvement is systematic. We depict on Fig. 3.1-left the inverse of ν and the optimal cost V_T (for $T = 15$ and $T = 50$) as functions of $\bar{D}$. We depict on Fig. 3.2-left the optimal cost as a function of T (for a given $\bar{D}$) which is decreasing accordingly to Lemma A2. We also compute the relative gain (for Prob. 1)

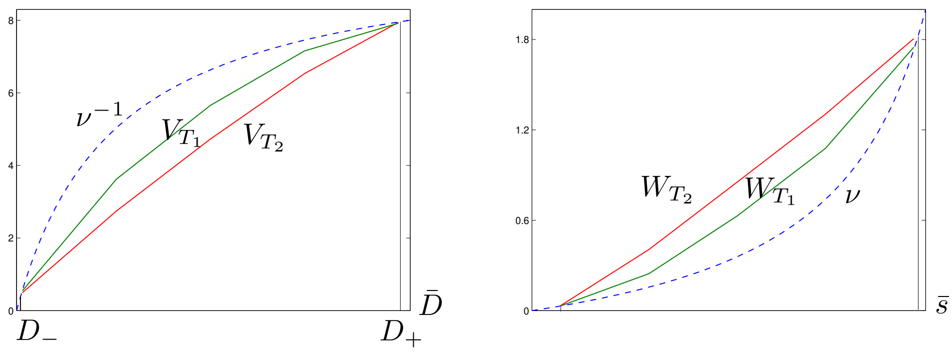


FIGURE 3.1 – *Left: V_{T_1}, V_{T_2}, ν^{-1} w.r.t. $\bar{D}$. Right: W_{T_1}, W_{T_2}, ν w.r.t. $\bar{s}$ (for the Contois kinetics with $T_1 = 15$, $T_2 = 50$, $s_{in} = 8$, $m = 2$, $K = 5$, $D_+ = 1.95$, $D_- = 0.02$)*

given by $G_1(T, \bar{D}) := (\bar{s} - V_T(\bar{D}))/\bar{s}$ using periodic control versus constant control and we depict the corresponding iso-values on Fig. 3.2-right. Such a diagram can help the practitioners to decide, depending on the characteristics of the application (nominal flow rate, maximal period on which average water quality can be considered), if a periodic operation is worth the ope-

rated one.
For Prob. 2, the constraint $\langle s \rangle_T = \bar{s}$ (on the state variable) is more delicate to handle. However, according to the duality given by Proposition 4.3, one can solve Prob. 1 for any $\bar{D} \in (D_-, D_+)$ and inverse the function V_T. Notice that the improvement condition implies that one has $W_T(\bar{s}) > \nu(\bar{s})$ in I. Therefore, in order to compute $W_T(\bar{s})$ for a given value $\bar{s} \in I$, one can look for $\bar{D}$ satisfying $V_T(\bar{D}) = \bar{s}$ only for values $\bar{D}$ above $\nu(\bar{s})$. We plot ν and the optimal cost W_T (for $T = 15$ and $T = 50$) as functions of $\bar{s}$ on Fig. 3.1-right. Similarly to Prob. 1, we compute the relative gain $G_2(T, \bar{D}) := (W_T(\bar{s}) - \bar{D})/\bar{D}$ and plot its iso-values on Fig. 3.3

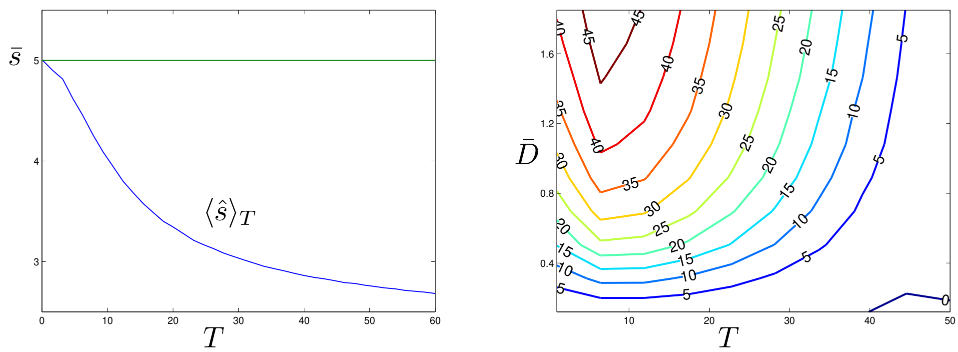


FIGURE 3.2 – *Left: optimal cost w.r.t. T for the Contois law with $s_{in} = 8$, $m = 2$, $K = 5$, $D_+ = 1.95$, $D_- = 0.02$, $\bar{D} = 0.5$. Right: Iso-values of G_1 (for the Contois function) in %.*

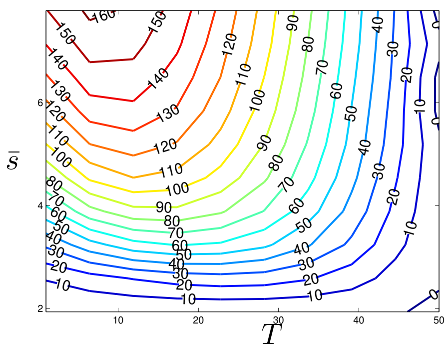


FIGURE 3.3 – *Iso-values of G_2 (for the Contois function) in %.*

For Hill's kinetics, we fix $\bar{D}$ in $(D_-, \mu(s_c))$ where the function is convex and chose $D_+ > \mu(s_c)$. Note that ν is increasing on $\mathbb{R}_+$, therefore, from Theorem A1, there exists a unique $(s_m, s_M) \in I^2$ satisfying (3.8)-(3.9) for any $T > 0$. We can then compute the cost $\langle s\rangle_T$ obtained with the control $\hat{D}_T$ (since it is defined by the pair (s_m, s_M)) that we denote $\hat{J}_T$ (see Fig. 3.4-left). It is proved in [BRTss] that the optimal cost is a decreasing function of T as long as periodic solutions belong to a domain where ν is convex (see Lemma A2). Numerical simulations give in our case a period $T_{max} \simeq 0.4$ for which the conditions of Theorem A3 are verified for any $T \leq T_{max}$. For $T > T_{max}$, we have no guarantee that bang-bang controls stay optimal and we ignore about the monotonicity of $T \mapsto \hat{J}_T$. Nevertheless, we observe on Fig. 3.4 that $T \mapsto \hat{J}_T$ still decreases on $(T_{max}, \bar{T})$, where $\bar{T}$ denotes the minimum of $T \mapsto \hat{J}_T$. We propose, for $T > \bar{T}$, a control function $\tilde{D}_T$ with $2k$ or $2(k+1)$ switches, where $k := E[T/\bar{T}] > 1$, as

$$\tilde{D}_T(t) := \begin{cases} \hat{D}_{\bar{T}}(t - i\bar{T}), \ t \in [i\bar{T}, (i+1)\bar{T}), \ i = \overline{0, k-1}, \\ \hat{D}_\tau(t - k\bar{T}), \ t \in [k\bar{T}, T) \text{ if } \tau = T - k\bar{T} > 0, \end{cases}$$

whose cost is given by

$$\tilde{J}_T := \frac{k\bar{T}\,\hat{J}_{\bar{T}} + \tau\,\hat{J}_\tau}{T} \quad (\text{with } \tau = T - k\bar{T}).$$

It is clear that $\tilde{J}_T = \hat{J}_{\bar{T}}$ when $T = k\bar{T}$ and is then below $\bar{s}$. Moreover, one has $\hat{J}_\tau < \bar{s}$ (as $\tau < \bar{T}$ and $\hat{J}_T < \bar{s}$ for $T \in (0, \bar{T}]$). We conclude that $\tilde{J}_T < \bar{s}$ when $\tau \neq 0$ (since it is a convex combination of $\hat{J}_{\bar{T}}$ and $\hat{J}_\tau$), see Fig. 3.4-left. We

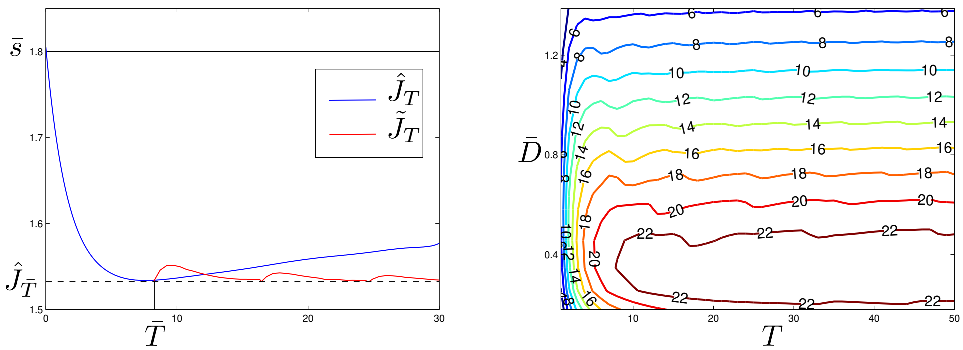


FIGURE 3.4 – *Left: costs $\hat{J}_T$, $\tilde{J}_T$ for the Hill law with $s_{in} = 6$, $m = 5$, $K = 3$, $n = 3$, $D_+ = 4.5$, $D_- = 0.05$, $\bar{s} = 1.8$, where $\bar{T} \simeq 8.21$. Right: Iso-values of $\tilde{G}_1$ (for the Hill function) in %.*

also plot the relative gain of control $\tilde{D}_T$ versus constant control, given by $\tilde{G}_1(T, \bar{D}) := (\bar{s} - \tilde{J}_T)/\bar{s}$ with $\bar{D} \in (D_-, \mu(s_c))$ (see Fig. 3.4-right). Finally, we present in Fig. 3.5 the value functions of Prob. 1 and Prob. 2, i.e., the maps

$\bar{D} \mapsto \tilde{V}_T(\bar{D})$, $\bar{s} \mapsto \tilde{W}_T(\bar{s})$, where $\tilde{V}_T(\bar{D})$ is the cost $\tilde{J}_T$ obtained for $\bar{D}$ and $\tilde{W}_T(\bar{s}) := \max\{\bar{D}\,;\,\tilde{V}_T(\bar{D})\}$.

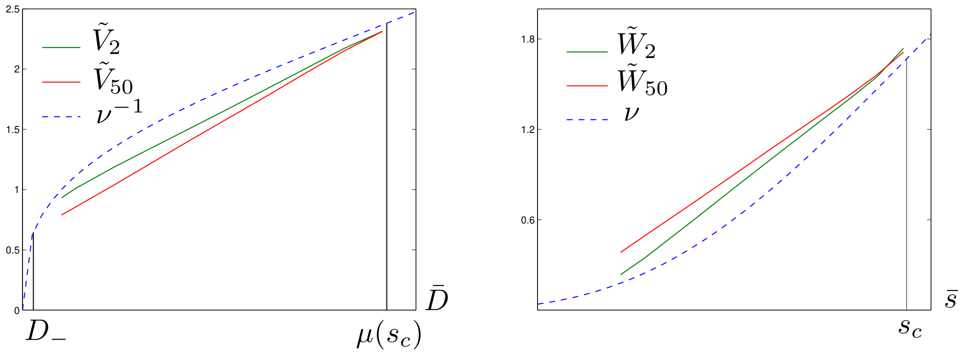


FIGURE 3.5 – *Left:* $\tilde{V}_{T_1}$, $\tilde{V}_{T_2}$, ν^{-1} *w.r.t.* $\bar{D}$. *Right:* $\tilde{W}_{T_1}$, $\tilde{W}_{T_2}$, ν *w.r.t.* $\bar{s}$ *(for Hill kinetics with* $T_1 = 2$, $T_2 = 50$, $s_{in} = 6$, $m = 5$, $K = 3$, $n = 3$ $D_+ = 4.5$, $D_- = 0.05$*)*

6 Conclusion

This work reveals the role played by the convexity of the growth function to obtain improvements with non-constant periodic controls, which allows to distinguish three possibilities: impossibility of improvement (Monod's or Haldane's kinetics), conditional improvement (Hill's kinetics) or systematic improvement (Contois's kinetics with $K < 1$). Thanks to a duality argumentation, we show that for both problems: minimizing the average output concentration under integral constraint on the control, or maximizing the integral of the control under constraint on the average output concentration, bang-bang controls are optimal among periodic solutions, and we characterize the two optimal switching times. This approach provides to practitioners the maximal improvement that can be expected playing with periodic operations. Further extensions of this work could consider multiple species (species coexistence in the chemostat being generically possible only for non-constant controls) or biogas production as an additional criterion.

7 Appendix

In this section, we recall the main results of [BRTss] (written with x in place of s). Consider a one-dimensional controlled dynamics

$$\dot{x} = f(x) + ug(x), \quad x \in \mathbb{R}, \quad u(\cdot) \in [-1, 1] \tag{3.12}$$

and the optimal control problem

$$\inf_{u \in \mathcal{U}} \frac{1}{T} \int_0^T \ell(x(t))\,\mathrm{d}t \text{ s.t. } x(0) = x(T) \text{ and } \langle u \rangle_T = \bar{u}$$

where $\mathcal{U}$ denotes the set of admissible controls. Functions f, g, ℓ are supposed to be of class C^1. In [BRTss], it is assumed that there is an interval $I := (a, b)$ with $a < b$ such that $g > 0$, $f - g < 0$ and $f + g > 0$ over I, with $f(a) - g(a) = 0$ and $f(b) + g(b) = 0$ (this amounts to require that I is invariant and that (3.12) is controllable).

Hypothesis 7.1. *The function $\psi := -f/g$ verifies*

(i) *There is a unique $\bar{x} \in I$ such that $\psi(\bar{x}) = \bar{u}$ and $(\psi(x) - \psi(\bar{x})(x - \bar{x}) > 0$ for any $x \in I \setminus \{\bar{x}\}$.*

(ii) *ℓ is increasing over I and $\gamma := \psi \circ \ell^{-1}$ is strictly convex increasing over $\ell(I)$.*

Next, define the function

$$\eta(x) := \frac{1}{f(x) + g(x)} - \frac{1}{f(x) - g(x)}, \quad x \in I,$$

and for x_m, x_M such that $a < x_m < \bar{x} < \bar{x}_M < b$ and $x(0) = \bar{x}$, consider the bang-bang control

$$\hat{u}_T(t) := \begin{cases} +1, \ 0 \le t < t_1 := \inf\{t > 0, \ x(t) = x_M\}, \\ -1, \ t_1 \le t < t_2 := \inf\{t > t_1, \ x(t) = x_m\}, \\ +1, \ t_2 \le t < T. \end{cases}$$

The following results are proved in [BRTss] (Theorem 3.6 and 4.1).

Theorem A1. *Under Hyp. 7.1*(i)*, for any* $T > 0$*, there exists a unique pair* $(x_m, x_M) \in I^2$ *such that*

$$\int_{x_m}^{x_M} \eta(x)\,\mathrm{d}x = T \text{ and } \int_{x_m}^{x_M} \psi(x)\eta(x)\,\mathrm{d}x = T\bar{u}. \tag{3.13}$$

Moreover, if Hyp. 7.1(ii) *is fulfilled then the control* $\hat{u}_T(\cdot)$ *defined by* (x_m, x_M) *satisfying* (3.13) *is optimal for the initial condition* $x(0) = \bar{x}$*.*

One has also the following property.

Lemma A2. *Under conditions of Theorem A1, the optimal cost (which corresponds to* $\hat{u}_T$*) is decreasing w.r.t.* T*.*

Hyp. 7.1 can be relaxed considering the interval $[x_T^-, x_T^+]$ with $x_T^- = x(t^+, u^+)$, $x_T^+ = x(t^-, u^-)$, where $x(\cdot, u^\pm)$ are the solutions of (3.12) for the one switch controls

$$u^-(t) = \left| \begin{array}{rl} -1, & t \in [0, t^-), \\ 1, & t \in [t^-, T], \end{array} \right. \quad ; \; u^+(t) = \left| \begin{array}{rl} 1, & t \in [0, t^+), \\ -1, & t \in [t^+, T], \end{array} \right.$$

with $x(0, u^\pm) = \bar{x}$ and t^-, t^+ are such that $x(T, u^\pm) = \bar{x}$.

Theorem A3. *For any* $T > 0$ *such that Hyp. 7.1*(i) *is fulfilled on* $[x_T^-, x_T^+]$ *instead of* I*, there exists a unique pair* $(x_m, x_M) \in I^2$ *satisfying* (3.13)*. If moreover Hyp. 7.1*(ii) *is fulfilled on* $[x_m, x_M]$*, then the control* $\hat{u}_T(\cdot)$ *is optimal.*

CHAPITRE 4

PERIODIC CONTROLS FOR DISCRIMINATING DENSITY DEPENDENT GROWTH IN THE CHEMOSTAT

Ce travail est à paraître en ligne dans les *proceedings of the* 58^{th} *IEEE Conference on Decision and Control*.

Contents

1 Introduction

Continuous culture has been invented simultaneously by Monod [Mon50] and Novick & Szilard [NS50b] in the fifties as a means to measure accurately growth rates of micro-organisms. Typically, the so-called *chemostat* device consists in feeding a culture vessel of volume V with a nutritive solution of concentration s_{in} at a constant flow rate Q, and then in extracting micro-organisms and substrate at the same rate Q from the culture vessel (so that the volume remains constant). The concentrations of micro-organisms and substrate, denoted respectively by b and s are then solutions of the following differential equations, that reflect the writing of the mass balance

$$\left\{ \begin{array}{rcl} \dot{s} &=& -\frac{1}{Y}\mu(s)b + D(s_{in} - s), \\ \dot{b} &=& \mu(s)b - Db, \end{array} \right. \tag{4.1}$$

where $D := Q/V$ denotes the dilution rate, Y the conversion rate and $\mu(\cdot)$ the specific growth rate. Each bacterial species is characterized by the positive constant Y and the non-negative function $s \mapsto \mu(s)$ satisfies $\mu(0) = 0$. The usual way to identify Y and $\mu(\cdot)$ are as follows:

1. The parameter Y is obtained from the *batch* mode which consists in taking $D = 0$ (see for instance [CCP+86]). One can check that solutions of (**??**) with positive initial conditions (s_0, b_0) converge asymptotically to the steady state $(s^\star, b^\star) := (0, b_0 + Yb_0)$. The parameter Y is thus determined by

$$Y = \frac{b^\star - b_0}{s_0}, s$$

which is usually considered as a robust method.

2. There are two general approaches for estimating the growth rate function $\mu(\cdot)$: either by adjusting the dynamical model to growth curves (in batch or in continuous culture) [BD90, Doc03], either by a series of steady-states. The first method is often considered less accurate because it deals with transient dynamics and biomass sampling can perturb the culture. The second method measures steady-states only, for which cell density no longer change with time (historically this method owes its name to "chemo-

stat"), and is considered more reliable [HH05]. The growth function $\mu(\cdot)$ is then identified step by step, considering a series of experiments with different constant values of D (see for instance [ZBG13]). One can easily check that positive equilibria $(s^\star, b^\star)$ of (4.1) are given by $\mu(s^\star) = D$ and $b^\star = Y(s_{in} - s^\star)$, when $s^\star$ is below s_{in}. Therefore each value of D provides a point $(s^\star, D)$ of the graph of $\mu(\cdot)$. This method implicitly assumes that there exists a unique solution s to the equation $\mu(s) = D$ and that the corresponding steady state is asymptotically stable. One can easily show (see for instance [HLRS17]) that this is fulfilled when $\mu(\cdot)$ is monotone and D is chosen less than $\mu(s_{in})$. For non-monotonic growth functions, other methods can be used (see for instance [SRRD13]).

Most often, the growth functions that suit data are concave increasing, such as the *Monod law* [Mon50]:

$$\mu_M(s) := \mu_{\max} \frac{s}{K_s + s}. \tag{4.2}$$

However, the function μ, differently to the parameter Y, might depend on environmental growth conditions (temperature, pH,...). Parameters μ_{max}, K_s could then depend on experiments and measurements. It also happens that the growth is inhibited by large concentrations b of the biomass, reflecting a *crowding effect* of micro-organisms. This is particularly met in bioreactors of industrial waste-water plants. Then, one has to look for *density dependent* functions, i.e., of the form $(s, b) \mapsto \mu(s, b)$, increasing with respect to s and non-increasing with respect to b. The most popular one is the *Contois law* [Con59], given by the following expression

$$\mu_C(s, b) := \mu_{\max} \frac{s}{s + kb}. \tag{4.3}$$

Notice that Contois law can be interpreted as a Monod expression but applied to the ratio s/b instead of s. However it does not generalize the Monod function for small density effect, but one can consider the *generalized Contois law*

$$\mu_{CG}(s, b) := \mu_{\max} \frac{s}{K_s + s + kb}, \tag{4.4}$$

to recover Monod law (taking $k = 0$) and Contois law (taking $K_s = 0$) expressions. Here the parameter $k > 0$ measures the density effect. Recently, another expression of density dependent function has been proposed and experimentally validated [KHT+18]:

$$\mu_K(s,b) := \mu_0 \frac{s}{b^\alpha}, \tag{4.5}$$

where α is a non-negative parameter.

Before setting up a model identification and control strategies, it is worth to mention that the choice between Monod law or density dependent extensions has important impacts on the predictions of the behavior of bacterial species (the greater the parameter k or α is, the greater the impacts are). First of all, steady state concentrations of substrate do not depend on the input concentration s_{in} in the case of Monod law, while it does in the case of Contois law [HLRS17]. This is clearly an issue for the piloting of waste-water plants for which the input concentration s_{in} fluctuates with time. Secondly, the Competitive Exclusion Principle [SW95, HLRS17] applies when the growth functions depend on s only: it asserts that only one species coexists in the long run in the chemostat model. On the contrary, a dependency on biomass concentration in the kinetics can allow coexistence of several species [LMR05, LAS06, LMR07], which can impact the performances of bioreactors. Therefore, the discrimination between substrate and ratio dependency is a question of prime importance, in particular for the choice of sensors and control laws.

The classical identification method we recalled previously imposes many experiments to reconstruct accurately the graph of μ (as one experiment provides only one point of its graph) and can be then lengthy and costly, whereas control laws do not necessarily require a perfect knowledge of the function μ (when it depends on s only, see for instance [BD90, RH02, AHSA03]). Notice that the identification of density dependent functions requires the additional measurement of the biomass, which is often more difficult and expensive to obtain than the abiotic component. Ratio-dependency has received a great attention in the ecological literature (see for instance [AG12]), and a conceptual experiment of several tanks in series has been designed to qualitatively discriminate ratio-dependency in the context of prey-predators [AS92]. Inspired by this approach, a similar experiment has been investigated in the context of bioreactors [HG07]. However, it imposes to be able to retain perfectly the biomass from one reactor to the following one, which can be a practical issue.

The purpose of the present work is to investigate a robust methodology for discriminating between the two kinds of growth functions, on the basis of a simple experiment with a single tank and the single measurement of the substrate. Instead of steady state operations, as in the methods described above, we propose to control the experiment with periodic dilution rate $D(\cdot)$ instead of constant ones. Periodic operations of chemostat have already been investigated in the literature, but for optimizing performances when the growth function is known (see for instance [AL87, AL89]).

The paper is organized as follows. In Section 2, we give results about periodic solutions of the chemostat model with periodic controls. In particular, we show in Proposition 2.1 the role played by the convexity of $\mu(\cdot)$ in the average value of $s(\cdot)$ over a period, compared to a steady state $s^{\star}$. This result is the core of the procedure that is presented in Section 3. In this section, we first define ”weak” and ”strong” density dependent effects, and we then give an experiment procedure based on bang-bang controls for discriminating between these two aforementioned cases with the single measurement of the variable s only. In Section 4, we justify the choice of bang-bang controls as the optimal ones. Finally, Section 5 illustrates this methodology on numerical simulations showing also the robustness of the method when environmental conditions could suddenly change.

2 The chemostat model under periodic control

In the sequel, we shall consider non-negative measurable control functions $D : \mathbb{R}_+ \to \mathbb{R}_+$ such that:
- the control $D(\cdot)$ is periodic of period T (to be chosen),
- the control $D(\cdot)$ is persistently exciting ([BD90]) and satisfies

$$\frac{1}{T}\int_0^T D(t)\,\mathrm{d}t = \bar{D} > 0, \tag{4.6}$$

where $\bar{D} > 0$ is given. Considering the total mass concentration $m = Ys + b$, (4.1) implies that

$$\dot{m} = D(t)(Ys_{in} - m).$$

Then, any periodic solution of (4.1) has to fulfill $m(t) = Ys_{in}$ for any time t. Therefore, we shall consider the reduced dynamics on the invariant line $Ys + b = Ys_{in}$, that is, the scalar dynamics

$$\dot{s} = F(s, D) := (s_{in} - s)(D - \nu(s)), \tag{4.7}$$

on the invariant interval $I := (0, s_{in})$, where the function ν is defined as follows

$$\nu(s) := \begin{cases} \mu(s) & \text{if } \mu = \mu_M, \\ \mu(s, Y(s_{in} - s)) & \text{if } \mu = \mu_{CG}. \end{cases}$$

Notice that in any case, the function ν is increasing. Therefore, for $\bar{D} < \nu(s_{in})$, there exists a unique solution $s^\star$ of

$$\nu(s^\star) = \bar{D},$$

with $x^\star := Y(s_{in} - s^\star) > 0$, and $(s^\star, x^\star)$ is thus the unique asymptotically stable equilibrium of (4.1) for the constant control $D = \bar{D}$. Let us define for convenience the function

$$\lambda(D) := \begin{cases} \nu^{-1}(D) & \text{if } D < \nu(s_{in}), \\ s_{in} & \text{otherwise.} \end{cases}$$

We start by giving a result about the convergence of solutions of (4.7) to a unique T-periodic solution, using the usual argumentation for scalar dynamics based on the Poincaré mapping (see for instance [SW95]).

Lemma 2.1. *Suppose that $\bar{D} < \nu(s_{in})$. Then, for any T-periodic control $D(\cdot)$ that satisfies* (4.6) *and any initial condition $s_0 \in I$, the associated solution of* (4.7) *converges asymptotically to the unique T-periodic solution of* (4.7) *associated with $D(\cdot)$.*

Proof. For a given T-periodic control $D(\cdot)$, consider the C^1 function $\phi : I \mapsto I$ defined by $\phi(s_0) := s(T, s_0)$ where $s(\cdot, s_0)$ denotes the solution of (4.7) for the initial condition s_0 at time 0. By differentiating ϕ, one has

$$\phi'(s_0) = \frac{\partial s}{\partial s_0}(T, s_0) = X(T),$$

where $X(\cdot)$ is the fundamental solution of the variational equation $\dot{z} = \frac{\partial f}{\partial s}(s(t, s_0), D(t))z$ with $z(0) = 1$. One gets:

$$\phi'(s_0) = e^{-\int_0^T [D(t) - \nu(s(t,s_0)) + \nu'(s(t,s_0))(s_{in} - s(t,s_0))]\, \mathrm{d}t}.$$

Clearly, $\phi' > 0$ and thus ϕ is increasing. For $s_0 = 0$, the solution cannot be identically equal to 0 because of condition (4.6) and $F(0, D) \geq 0$. Therefore one has $\phi(0) = s(0, T) > 0$. For $s_0 = s_{in}$, the solution of (4.7) is constant equal to s_{in}. Then $\phi(s_{in}) = s_{in}$ and

$$\phi'(s_{in}) = e^{-\int_0^T [D(t) - \nu(s_{in})]\, \mathrm{d}t}.$$

Since $\bar{D} < \nu(s_{in})$, we deduce that $\phi'(s_{in}) > 1$. Consequently, the C^1 function $s_0 \mapsto \psi(s_0) := \phi(s_0) - s_0$ satisfies $\psi(0) > 0$, $\psi(s_{in}) = 0$, and $\psi'(s_{in}) > 0$, thus it has necessarily a zero in I which proves the existence of a fixed point $\bar{s}_0$ of ϕ as well as the existence of a T-periodic solution $\bar{s}(\cdot) := s(\cdot, \bar{s}_0)$ of (4.7). Moreover, at $s_0 = \bar{s}_0$, one finds

$$\psi'(\bar{s}_0) = e^{-\int_0^T [D(t) - \nu(\bar{s}(t)) + \nu'(\bar{s}(t))(s_{in} - \bar{s}(t)]\, \mathrm{d}t} - 1.$$

Observe now that for a periodic solution $\bar{s}(\cdot)$, one has

$$0 = \ln\left(\frac{s_{in} - \bar{s}(T)}{s_{in} - \bar{s}_0}\right) = -\int_0^T [D(t) - \nu(\bar{s}(t))]\, \mathrm{d}t, \tag{4.8}$$

from which one obtains

$$\psi'(\bar{s}_0) = e^{-\int_0^T \nu'(\bar{s}(t))(s_{in} - \bar{s}(t))\, \mathrm{d}t} - 1 < 0.$$

Since at every point $\bar{s}_0 \in I$ such that $\psi(s_0) = 0$, one has $\psi'(\bar{s}_0) < 0$, we then conclude that ψ has exactly one zero $\bar{s}_0 \in I$ which shows the uniqueness of the T-periodic solution $\bar{s}(\cdot)$.

Notice that $\bar{s}(\cdot)$ is asymptotically stable since $\phi'(\bar{s}_0) < 1$ (see [SW95]).

Finally, for any $s_0 \in I \setminus \{\bar{s}_0\}$, the sequence $(\phi^n(s_0))_n$ is monotone since ϕ is increasing, and thus it converges to a fixed point of ϕ. The only fixed point at boundary of I is s_{in} with $\phi'(s_{in}) < 1$ which implies that $\phi(s_0) < s_0$ for s_0 close to s_{in}. One can verify that $\phi^n(s_0)$ necessarily converges to $\bar{s}_0$ when n goes to infinity which implies that $|s(t, s_0) - \bar{s}(t)| \to 0$ when t goes to infinity. □

The next result will play an important role in the following. The key point is to consider functions $D(\cdot)$ that fulfill (4.6) for the same value $\bar{D}$. Hereafter, for a given non constant T-periodic control $D(\cdot)$ that fulfills (4.6), we denote by $\bar{s}(\cdot)$ the unique periodic solution associated with $D(\cdot)$.

Proposition 2.1. *If the restriction of $\nu(\cdot)$ to the interval $J := \{\nu(\bar{s}(t)),\ t \in [0, T]\}$ is convex, respectively concave, and does not coincide with a linear function on this interval, then*

$$\hat{s} := \frac{1}{T}\int_0^T \bar{s}(t)\,\mathrm{d}t\ < s^\star, \quad \textit{respectively}\ \hat{s} := \frac{1}{T}\int_0^T \bar{s}(t)\,\mathrm{d}t\ > s^\star.$$

Proof. Let $\bar{s}(\cdot)$ be a non-constant T-periodic solution with values in I. From (4.8), we deduce the equality

$$\frac{1}{T}\int_0^T \nu(\bar{s}(t))\,\mathrm{d}t = \bar{D}. \tag{4.9}$$

When ν is convex on the interval J, applying Jensen's inequality to ν yields

$$\nu\left(\frac{1}{T}\int_0^T \bar{s}(t)\,\mathrm{d}t\right) \leq \frac{1}{T}\int_0^T \nu(\bar{s}(t))\,\mathrm{d}t. \tag{4.10}$$

Similarly, when ν is concave on J, one obtains the reversed inequality. Since equality in Jensen's inequality holds true if and only if ν is affine over J, the inequality in (4.10) is strict. Combining (4.9)-(4.10) then gives

$$\nu\left(\frac{1}{T}\int_0^T \bar{s}(t)\,\mathrm{d}t\right) < \nu(\bar{s}),$$

and, as ν is increasing, we get

$$\frac{1}{T}\int_0^T \bar{s}(t)\,\mathrm{d}t < s^\star.$$

The reversed inequality is obtained in the same way when v is concave over J. □

Lemma 2.2. *The solution $\bar{s}$ verifies the inequalities:*

$$\min_{t\in[0,T]} \bar{s}(t) \leq s^\star \leq \max_{t\in[0,T]} \bar{s}(t).$$

Proof. Equality (4.9) can be equivalently written

$$\frac{1}{T}\int_0^T [\nu(\bar{s}(t)) - \nu(s^\star)]\,\mathrm{d}t = 0.$$

Then, the function $t \mapsto \nu(\bar{s}(t)) - \nu(s^\star)$ has to change its sign over $[0, T]$. Since ν is monotone over I, we deduce that there exists t_- and t_+ in $[0, T]$ such that $\bar{s}(t_-) < s^\star < \bar{s}(t_+)$. □

Without any loss of generality, we can then consider periodic solutions of (4.7) with periodic controls satisfying (4.6) and such that $\bar{s}(0) = s^\star$. The constant solution $s = s^\star$ for constant $D = \bar{D}$ is one of them. We show how to use Proposition 2.1 to discriminate density-dependency.

3 A discriminating procedure

Consider the growth functions ν associated to Monod and (generalized) Contois laws (following the notation of Section 2):

$$\nu_M(s) := \mu_{max}\frac{s}{K_s + s}\ ;\ \nu_{CG}(s) := \mu_{max}\frac{s}{K_s + s + kY(s_{in} - s)}.$$

One can easily check that these functions are increasing. A straightforward computation also gives

$$\nu_M''(s) = \frac{-2K_s\mu_{max}}{(K_s + s)^3}\ ;\ \nu_{CG}''(s) = \mu_{max}\frac{2(kY - 1)(K_s + kYs_{in})}{(K_s + s + kY(s_{in} - s))^3}.$$

One can observe that ν_M is always concave, while ν_{CG} is convex when $kY > 1$. We shall say that the growth rate ν has a *weak (or null) density* *effect* if it follows the laws of Monod or (generalized) Contois with $k \leq 1/Y$. Conversely, we shall say that it has a *strong density effect* if it follows the law of (generalized) Contois with $k > 1/Y$.

We investigate now how to use the results of Proposition 2.1 to discriminate between convex and concave functions using periodic controls. Let us underline that for a given value of $s^\star$, determining periodic non constant functions $D(\cdot)$ such that the solution $s(\cdot)$ with $s(0) = s^\star$ is periodic requires the perfect knowledge of the function μ and the constants Y and s_{in}. On another hand, waiting for the convergence to a periodic solution could be long and difficult to test in practice. We propose below an experiment procedure to discriminate between weak and strong density effects.

Procedure . *It consists in conducting a single chemostat experiment with three phases: constant, bang-bang and constant. It requires the single measurement of the substrate s.*

Phase 1. Conduct an experiment for a constant dilution rate D with $0 < D < \nu(s_{in})$ (to avoid the washout equilibrium) and wait the chemostat to be at (quasi) steady state. Let s^{eq} be the measurement of s, and t_0 the current time.

Phase 2. Choose two values D_{min}, D_{max} such that $0 \leq D_{min} < D < D_{max}$, and two time durations $\delta_1 > 0$, $\delta_2 > 0$. Apply first $D = D_{max}$ on the time interval $[t_0, t_0 + \delta_1)$. Next, apply $D = D_{min}$ until the time $\bar{t}$ such that $s(\bar{t}) = s_{eq}$, and carry on applying $D = D_{min}$ during the duration δ_2. Finally, change the flow rate value to $D = D_{max}$ for $t > \bar{t} + \delta_2$. Store the measurements history $\{s(t)\}_{t \in [t_0, t_0+T]}$ until the first time $t_0 + T > \bar{t} + \delta_2$ such that $s(t_0 + T) = s^{eq}$ (which exists as $0 \leq D_{min} < D < D_{max}$). To summarize, one applies

$$D(t) := \begin{cases} D_{\max}, & t \in [t_0, t_0 + \delta_1), \\ D_{min}, & t \in [t_0 + \delta_1, \bar{t} + \delta_2), \\ D_{max}, & t \in [\bar{t} + \delta_2, t_0 + T]. \end{cases} \tag{4.11}$$

Let $\bar{D}$ be the mean value of the dilution rate over $[t_0, t_0 + T]$:

$$\bar{D} := \frac{D_{max}(T + t_0 - \bar{t} + \delta_1 - \delta_2) + D_{min}(\bar{t} - t_0 + \delta_2 - \delta_1)}{T}.$$

Phase 3. *Apply for $t > t_0 + T$ the constant dilution rate $D := \bar{D}$ and wait for the (quasi) steady state, denoted by $s^\star$. With the data stored on $[t_0, t_0 + T]$, determine the average value of s on $[t_0, t_0 + T]$:*

$$\hat{s} := \frac{1}{T} \int_{t_0}^{t_0+T} s(t)\,\mathrm{d}t.$$

Conclusion. *If $\hat{s} < s^\star$, we validate the strong density effect.*

Phase 1 allows the chemostat to be at steady state, so that dynamics (4.7) is considered to be valid at time t_0. The phase 2 generates a T-periodic solution with a periodic control, over a single period, without having to wait the convergence to the periodic solution. Phase 3 compares the average value $\hat{s}$ of the substrate concentration on the periodic solution with the steady state one $s^\star$ obtained for a constant control $\bar{D}$ equal to the average value of the periodic control. Proposition (2.1) allows to conclude.

Remark 3.1. *Note that $s^\star$ has no a priori reason to be equal to s^{eq}, as the initial value of D is arbitrary.*

Remark 3.2. *This procedure is a way to discriminate between Monod and (generalized) Contois laws. It allows also to discriminate between convex or concave functions on the invariant interval $(\lambda(D_{min}), \lambda(D_{max}))$ and can then be applied to other situations whose growth is expected to follow expressions such as* (4.5). *For this function, one has*

$$\nu_K''(s) = \alpha\mu_0 \frac{2s_{in} + (\alpha - 1)s}{(s_{in} - s)^{\alpha+2}}.$$

Therefore, ν_K is convex when α is larger than one (which has been observed in [KHT+18]) or when s_{in} is large enough.

The proposed procedure considers bang-bang periodic controls for discriminating convex functions ν. One may wonder if the periodic control that satisfies (4.6) allows to obtain the largest difference between $\hat{s}$ and $s^\star$. We justify this choice in the next section.

4 The best shape of periodic controls

The goal of this section is to justify the construction of D in (4.11) as a bang-bang control, in terms of optimal control. Given bounds D_{min}, D_{max} on the control $D(\cdot)$ with $0 \leq D_{min} < \bar{D} < D_{max}$, and given a convex function ν, we look for T-periodic controls $D(\cdot)$ that satisfy (4.6) with

$$D(t) \in [D_{min}, D_{max}] \quad \text{a.e. } t \geq 0, \tag{4.12}$$

such that the associated solution $\bar{s}$ with $\bar{s}(0) = s^\star$ is T-periodic and gives the smallest average value $\hat{s}$:

$$\min_{D(\cdot)} \hat{s} := \frac{1}{T}\int_0^T s(t)\,\mathrm{d}t. \tag{4.13}$$

Similarly, when ν is concave, we would look for controls $D(\cdot)$ such that $\hat{s}$ is the largest. Problem (4.13) is an optimal periodic control problem. Several contributions about the theory of optimal control with periodic controls are available in the literature (see for instance [GLR74, Gil77, WS90] and references herein). However, in the present case, it has to fulfill the additional integral constraint (4.6), which has not been yet studied, up to our knowledge, expect in the recent works [BRT18, BRTss].

Let $K := (\lambda(D_{min}), \lambda(D_{max}))$ and introduce the function $\eta : K \to \mathbb{R}_+^*$ as

$$\eta(s) := \frac{1}{(s_{in} - s)(D_{max} - \nu(s))(\nu(s) - D_{min})}, \quad s \in K.$$

One has the following optimality result.

Proposition 4.1. *Assume ν to be convex increasing on K, and take arbitrarily $\bar{D} \in (D_{min}, D_{max})$ and $T > 0$. Then:*

(i) *There are exactly two T-periodic optimal solutions of* (4.13) *under the constraints* (4.6) *and* (4.12), *and such that $s(0) = s^\star$.*

(i) *The controls $D^{\pm}$ generate the two optimal trajectories (with same cost):*

$$D^{+}(t) := \begin{cases} D_{max}, & t \in [0, t_1^{+}) \cup [t_2^{+}, T) \text{ mod } T, \\ D_{min}, & t \in [t_1^{+}, t_2^{+}) \text{ mod } T, \end{cases} \tag{4.14}$$

$$D^{-}(t) := \begin{cases} D_{min}, & t \in [0, t_1^{-}) \cup [t_2^{-}, T) \text{ mod } T, \\ D_{max}, & t \in [t_1^{-}, t_2 -) \text{ mod } T, \end{cases}$$

where times $t_i^{\pm}$, $i = 1, 2$ are given by:

$$\begin{array}{ll} t_1^{+} := \inf\{t \geq 0 \; ; \; s(t) = s_M\}, & t_2^{+} := \inf\{t \geq t_1^{+} \; ; \; s(t) = s_m\}, \\ t_1^{-} := \inf\{t \geq 0 \; ; \; s(t) = s_m\}, & t_2^{-} := \inf\{t \geq t_1^{-} \; ; \; s(t) = s_M\}, \end{array} \tag{4.15}$$

and where (s_m, s_M) is the unique pair in K^2 that satisfies

$$\int_{s_m}^{s_M} \eta(s) \, \mathrm{d}s = \frac{T}{D_{max} - D_{min}}, \int_{s_m}^{s_M} \eta(s)\nu(s) \, \mathrm{d}s = \frac{\bar{D}T}{D_{max} - D_{min}}. \tag{4.16}$$

Proof. We apply a result of [BRTss] that we recall in the Appendix 7. One can easily check that (4.13) is a particular instance of (4.18) taking $I' = K$, $\tilde{\eta} = \eta$, $f(s) = (s_{in} - s)\left[\frac{D_{max}+D_{min}}{2} - \nu(s)\right]$, $g(s) = (s_{in} - s)\frac{D_{max}-D_{min}}{2}$ and $\ell(s) = s$, and verify that the required assumptions to apply the result of the Appendix 7 are satisfied. The optimal solutions $u^{\pm}$ of (4.18) provided by (4.20) once translated into (4.13), lead to the optimal solutions $D^{\pm}$ of (4.13) given by (4.14)-(4.15). As well, (4.16) straightforwardly follows from (4.19). □

This result gives the existence of δ_1, δ_2 that give the largest value $|\hat{s} - s^{\star}|$ in the procedure of Section 3. Let us underline that $s^{\star}$ is not known in advance (as the function ν is unknown) but is learned in phase 2 of the procedure.

5 Numerical illustrations

We have considered Monod and Contois laws (see the graph of the associated functions ν on Fig. 4.1) that provide (for the same values of D and s_{in}) the same steady state $s^{eq} = 1$. It is then not possible to distinguish between these two functions without changing the value of D. Fig. 4.2 illustrates the three phases generated by the procedure (phase 1: blue, phase 2: green, phase 3: red) with the average value $\hat{s}$ (in black).

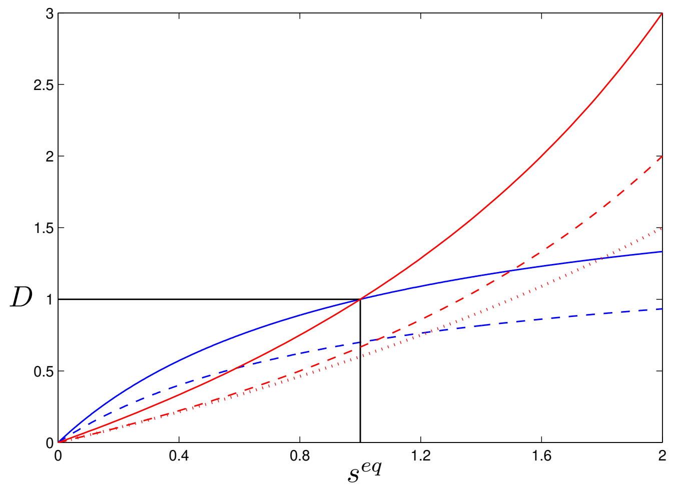


FIGURE 4.1 – *In blue: Monod's kinetics for $K_s = 2$; $Y = 1$; $s_{in} = 2$; $\mu_{max} = 2$ (plain) and $\mu_{max} = 1.4$ (dashed). In red: Contois's kinetics for $K_s = 0$; $Y = 1$; $k = 2$; $s_{in} = 2$; $\mu_{max} = 3$ (plain); $\mu_{max} = 2$ (dashed) and $s_{in} = 3$ (dotted).*

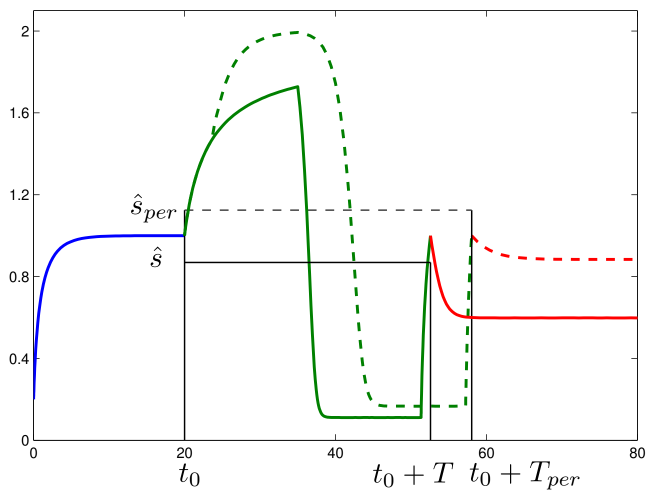


FIGURE 4.2 – *Solution $s(\cdot)$ generated by the procedure (for the Monod law) with $s(0) = 0.2$; $K_s = 1$; $\mu_{max} = 2$; $Y = 1$; $s_{in} = 2$; $D_{max} = 1.3$; $D_{min} = 0.2$ (in dashed under the perturbation)*

Then Tables 4.1 and 4.2 (left columns) gives the difference $\Delta s := \hat{s} - s^\star$ provided by the procedure, for different values of the durations δ_1, δ_2. One can see that a larger period T does not necessarily gives a larger difference

Δs. One has to choose larger values for both δ_1 and δ_2 to increase the sensitivity of the test.

	Without perturbation	Perturbation on s_{in}	Perturbation on μ_{max}
	$s_{in} = 2$, $\mu_{max} = 2$	$s_{in} = 2$ and 3, $\mu_{max} = 2$	$s_{in} = 2$, $\mu_{max} = 2$ and 1.4
$\delta_1 = 10$, $\delta_2 = 8$	$\Delta s = 0.24$ $T = 20.4$ $\bar{D} = 0.81$	$\Delta s = 0.24$ $T = 19.2$ $\bar{D} = 0.81$	$\Delta s = 0.05$ $T = 23$ $\bar{D} = 0.71$
$\delta_1 = 15$, $\delta_2 = 8$	$\Delta s = 0.26$ $T = 25.6$ $\bar{D} = 0.90$	$\Delta s = 0.28$ $T = 24.2$ $\bar{D} = 0.91$	$\Delta s = 0.15$ $T = 31$ $\bar{D} = 0.76$
$\delta_1 = 10$, $\delta_2 = 15$	$\Delta s = 0.23$ $T = 27.4$ $\bar{D} = 0.65$	$\Delta s = 0.23$ $T = 26.2$ $\bar{D} = 0.65$	$\Delta s = 0.14$ $T = 30$ $\bar{D} = 0.59$
$\delta_1 = 15$, $\delta_2 = 15$	$\Delta s = 0.27$ $T = 32.6$ $\bar{D} = 0.75$	$\Delta s = 0.29$ $T = 31.3$ $\bar{D} = 0.76$	$\Delta s = 0.24$ $T = 38$ $\bar{D} = 0.66$

TABLE 4.1 – *Results for Monod law:* $K_S = 2$; $Y = 1$; $D_{max} = 1.3$; $D_{min} = 0.2$.

	Without perturbation:	Perturbation on s_{in}:	Perturbation on μ_{max}:
	$s_{in} = 2$, $\mu_{max} = 3$	$s_{in} = 2$ and 3, $\mu_{max} = 3$	$s_{in} = 2$, $\mu_{max} = 3$ and 2
$\delta_1 = 10$, $\delta_2 = 8$	$\Delta s = -0.20$ $T = 19.5$ $\bar{D} = 1.47$	$\Delta s = -0.55$ $T = 19.73$ $\bar{D} = 1.44$	$\Delta s = -0.32$ $T = 22.1$ $\bar{D} = 1.31$
$\delta_1 = 15$, $\delta_2 = 8$	$\Delta s = -0.18$ $T = 24.5$ $\bar{D} = 1.72$	$\Delta s = -0.45$ $T = 24.8$ $\bar{D} = 1.69$	$\Delta s = -0.26$ $T = 29.2$ $\bar{D} = 1.46$
$\delta_1 = 10$, $\delta_2 = 15$	$\Delta s = -0.22$ $T = 26.5$ $\bar{D} = 1.11$	$\Delta s = -0.51$ $T = 26.73$ $\bar{D} = 1.09$	$\Delta s = -0.35$ $T = 29.1$ $\bar{D} = 1.02$
$\delta_1 = 15$, $\delta_2 = 15$	$\Delta s = -0.23$ $T = 31.5$ $\bar{D} = 1.36$	$\Delta s = -0.48$ $T = 31.8$ $\bar{D} = 1.34$	$\Delta s = -0.31$ $T = 36.2$ $\bar{D} = 1.20$

TABLE 4.2 – *Results for Contois law:* $K_s = 0$;$k = 2$; *Y=1;* $D_{max} = 2.7$; $D_{min} = 0.1$.

The interest of the procedure, compared to classical growth identification, relies on its robustness with respect to disturbances. It is well known that continuous cultures are often subject to uncontrolled environmental changes, such as variations of s_{in}, temperature or pH which directly impact the maximum growth rate $\mu_{\max}$. We have simulated such a sudden change during the phase 2 of the procedure. These changes correspond to the functions ν in dashed line on Fig. 4.1 and modify the s-trajectory as depicted in dashed line on Fig. 4.2. Tables 4.1 and 4.2 present the results under two kinds of disturbances: change of s_{in} and change of μ_{max}. In any case, one can observe that the sign of Δs is preserved. Notice that under these changes, the duration T is modified and the s-solution between t_0 and $t_0 + T$ is no longer periodic, neither for the old or the new functions ν, but Δs is a continuous function of the parameters s_{in}, μ_{max}, justifying the robustness of the test.

6 Conclusion

We have proposed an experiment procedure to test a density dependency in the chemostat, using non-constant dilution rates. This procedure consists in a sequence of a steady state, a periodic operation over only one period, and a last steady-state, with the single measurement of the substrate concentration. We have shown its robustness with respect to disturbances on the growth curves during the procedure, demonstrating its interest compared to classical identification methods. For more complex growths for which the function μ could be neither convex or concave on their whole domain (such as the Hill function [Mos58] or the one from the Microbial Transition State theory [DLB14]), one would need to choose adequately the bounds D_{min}, D_{max} in the procedure. This is an open problem that could be the matter of a future work.

7 Appendix

We recall results from [BRTss] about optimal periodic control. Consider the following dynamical system:

$$\dot{x} = f(x) + u(t)g(x),\ u(t) \in [-1, 1] \text{ and } x(0) = \bar{x}, \tag{4.17}$$

where $f : I' \to \mathbb{R}$, $g : I' \to \mathbb{R}_+^*$ and I' is an open interval of $\mathbb{R}$. f and g are assumed to be C^1 with $f + g > 0$ and $f - g < 0$ over I'. Consider now the following optimal control problem:

$$\min_{u(\cdot)} \frac{1}{T} \int_0^T \ell(x(t))\, \mathrm{d}t, \tag{4.18}$$

where $\ell : I' \to \mathbb{R}$ and $x(\cdot)$ is a T-periodic solution of (4.17) associated with a measurable control $u(\cdot)$ satisfying

$$\frac{1}{T} \int_0^T u(t)\, \mathrm{d}t = \bar{u},$$

where $\bar{u} := -f(\bar{x})/g(\bar{x})$. Define $\psi : I' \to \mathbb{R}$ as $\psi := -\frac{f}{g}$ and $\gamma := \psi \circ \ell^{-1}$ whenever ℓ^{-1} is well-defined, as well as

$$\tilde{\eta}(x) := \frac{1}{f(x) + g(x)} - \frac{1}{f(x) - g(x)},\ x \in I'.$$

It is supposed in [BRTss] that I' is invariant by the dynamics (4.17) with $\psi(I') \subset [-1, 1]$, and that there is a unique $\bar{x} \in I'$ s.t.

$$(\psi(x) - \psi(\bar{x}))(x - \bar{x}) > 0,\ x \in I',\ x \neq \bar{x}.$$

It is also shown in that when ℓ is increasing and γ strictly convex increasing, there is a unique $(x_m, x_M) \in I'^2$ s.t.

$$\int_{x_m}^{x_M} \tilde{\eta}(x)\, \mathrm{d}x = T, \quad \int_{x_m}^{x_M} \tilde{\eta}(x)\psi(x)\, \mathrm{d}x = T\bar{u}. \tag{4.19}$$

Theorem 3.6 in [BRTss] shows that the bang-bang control whose restriction over $[0, T]$ is defined as

$$u^+(t) := \left| \begin{array}{lll} +1 & \text{if} & t \in [0, \tilde{t}_1], \\ -1 & \text{if} & t \in [\tilde{t}_1, \tilde{t}_2], \\ +1 & \text{if} & t \in [\tilde{t}_2, T], \end{array} \right. \tag{4.20}$$

(where $\tilde{t}_1$ is the first time x reaches x_M and $\tilde{t}_2$ the first time $t > \tilde{t}_1$ where x reaches x_m) is optimal (up to a unique time translation defining an optimal control u^- with same cost).

CHAPITRE 5

WEAK RESILIENCE OF THE CHEMOSTAT MODEL TO A SPECIES INVASION WITH NON-AUTONOMOUS REMOVAL RATES

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Contents

1 Introduction

The chemostat model describes microbial ecosystems which are continuously fed by nutrients, as it can be found in natural environments, such as lakes, lagoons, wetlands... and in experimental or industrial bioreactors. Being open systems, chemostats are naturally subject to external perturbations such as species invasion.

In this paper, we focus on the classical chemostat model, for which the Competitive Exclusion Principle (CEP) holds [AM80]. For a single limiting resource, the CEP states that no more than one species (generically) survives in the long term under constant fed conditions (input substrate concentration and flow rate of the incoming resource), see, *e.g.*, [HLRS17, SW95].

For dynamical systems, *resilience* is often described as the ability of a system to return to an original state (typically a steady state) after a transient perturbation [MDC11]. In the present work, we wish to study the resilience of the chemostat model to invasions by other species considered as disturbances. In particular, we are interested in the possibility of extending the resilience domain using time-varying input conditions.

For the chemostat model, it has already been pointed out that non-constant removal rates could allow the coexistence of two species, under some precise integral conditions, see for instance [Smi81, BHW85, LP95, PH00, NT05, WP08] for periodic removal rates or [LRS09] for slow varying environments. The idea is to create a time-varying growth environment which alternates the favored species. Recently, the question of quantifying the excursions of the state variables in the chemostat model under periodic removal rates, has been studied in [BRT18, BRTss], but for the mono-specific case only. To our knowledge, it has not been investigated how to synthesize time-varying removal rates allowing resilience of a mono-specific chemostat system in presence of an invasion by a new species, in such a way that the resident species returns to the same density level than before invasion an infinite number of times. The design of such time-varying removal rate is precisely the matter of the present work.

The paper is structured as follows. In Section 2, we state the resilience problem in the context of the chemostat model with two species. In particular, we show the existence of a threshold on the level of the resident species above which resilience is lost that allows us to introduce a concept of *weak resilience* in a time varying context. In Section 3, we provide a construction of a time-varying removal rate which guarantees weak resilience in the sense given in Section 2. In Section 4, we show that there exist weakly resilient periodic solutions, and conjecture that there exists an unique periodic solution associated to the time-varying removal rate that we construct. Then, we show that any solution of the system associated with this time-varying removal rate converges to this periodic solution.

2 Assumptions and definition of weak resilience

We start by recalling the chemostat model with two microbial species of concentrations x_1, x_2, respectively, that compete for a single resource of concentration s:

$$\begin{cases} \dot{s} = -\mu_1(s)x_1 - \mu_2(s)x_2 + D(s_{in} - s), \\ \dot{x_1} = (\mu_1(s) - D)x_1, \\ \dot{x_2} = (\mu_2(s) - D)x_2, \end{cases} \tag{5.1}$$

in which the yield coefficients are equal to one. Parameters D and s_{in} are respectively the removal rate (imposed by the input flow rate) and the input concentration of the resource. Here, species 1 and 2 play the respective roles of the *resident* and *invasive* species, as described in the introduction: the chemostat is considered first with the species 1 (the resident) alone and at some time t_0 (chosen equal to 0 for simplicity), the invasive species 2 appears. In the sequel, we consider that the following assumption on the growth functions $\mu_i(\cdot)$, $i = 1, 2$ in (5.1) is fulfilled.

Assumption 2.1. *The functions $\mu_i(\cdot)$ are of class C^1, monotone increasing with $\mu_i(0) = 0$, $i = 1, 2$.*

For $i = 1, 2$, the *break-even concentration* for species i (related to the parameter s_{in}) is defined as

$$\lambda_i(D) := \sup \{s \in [0, s_{in}] \ ; \ \mu_i(s) < D\} \in [0, +\infty], \quad i = 1, 2.$$

When $\lambda_i(D) < +\infty$, and because μ_i are strictly monotone, one has

$$\mu_i(\lambda_i(D)) = D \quad \Longleftrightarrow \quad \lambda_i(D) = \mu_i^{-1}(D).$$

Consider now the competition between the two species. The CEP states that only the species that realizes the minimum of the numbers $\lambda_i(D)$, $i = 1, 2$ (with $D > 0$) has a non null concentration at steady state. For a given constant D with $0 < D < \mu_1(s_{in})$, this means that species 1 is excluded by an invasion by a species 2 when $\lambda_2(D) < \lambda_1(D)$, or equivalently that the state of (5.1) converges asymptotically to $(\lambda_2(D), 0, s_{in} - \lambda_2(D))$. Let us now consider a situation for which the dominance of one growth function over the other one is alternated with respect to the level of the resource, in the following way.

Assumption 2.2. *There exists* $\bar{s} \in (0, s_{in})$ *such that*

$$(\mu_1(s) - \mu_2(s))(s - \bar{s}) > 0, \quad s \in [0, s_{in}] \text{ and } s \neq \bar{s}. \tag{5.2}$$

In the rest of the paper, we assume that this assumption holds true. Note that for Monod's growth functions [Mon50] (which are quite popular in microbiology and bio-processes), one has

$$\mu_i(s) = \frac{\bar{\mu}_i s}{K_i + s}, \quad i = 1, 2.$$

Assumption 2.2 then amounts to have the following condition

$$0 < \bar{\mu}_2 K_1 - \bar{\mu}_1 K_2 < (\bar{\mu}_1 - \bar{\mu}_2) s_{in}$$

to be fulfilled. From (5.2), one must have $\mu_1(\bar{s}) - \mu_2(\bar{s}) = 0$, therefore we set

$$\bar{D} := \mu_1(\bar{s}) = \mu_2(\bar{s}).$$

We formulate the problem of invasion of species 1 by species 2 as follows. If only species 1 is present in a bioreactor (*i.e.*, $x_2 = 0$ in (5.1)) then, for a constant value of D, a straightforward analysis shows that a necessary condition for species 1 to survive is to have $\lambda_1(D) < s_{in}$. The corresponding steady state is then given by $(\lambda_1(D), s_{in} - \lambda_1(D))$, which is globally asymptotically stable in the (s, x_1) positive plane (see for instance [HLRS17]). Moreover, if D is less than $\bar{D}$, then the concentration of species 1 at steady state, $x_1^{eq} := s_{in} - \lambda_1(D))$, is necessarily such that $x_1^{eq} \geq \bar{x}_1$ with

$$\bar{x}_1 := s_{in} - \mu_1^{-1}(\bar{D}) = s_{in} - \bar{s} > 0.$$

Suppose now that a new species (species 2) that fulfills Assumption 2.2 invades the growth vessel and that the removal rate D is less than $\bar{D}$. In that case, the state of (5.1) converges asymptotically to $(\lambda_2(D), 0, s_{in} - \lambda_2(D))$, that is the concentration of species 1 converges to 0, contrary to the case when it is alone. Given a threshold $\bar{x}_1 > 0$, we shall say that the dynamics is *not resilient* to invasion when the solutions of the system do not converge to the set $\{x_1 \geq \bar{x}_1\}$. The question of interest is to investigate if, even so, the system can be resilient with x_1 above the threshold $\bar{x}_1$, when considering time-varying removal rate $D(\cdot)$. First, one can easily check that the domain

$$\mathcal{I} := \{(s, x_1, x_2) \in \mathbb{R}_+^3 \; ; \; s + x_1 + x_2 = s_{in}\},$$

is an invariant and attractive set for the dynamics (5.1) when $D(\cdot)$ is persistently exciting . We shall consider in the sequel the reduced dynamics on this domain, that is,

$$\dot{x} = f(x, D) := \begin{bmatrix} (\mu_1(s_{in} - x_1 - x_2) - D)x_1 \\ (\mu_2(s_{in} - x_1 - x_2) - D)x_2 \end{bmatrix}, \tag{5.3}$$

defined on the invariant set

$$\mathcal{S} := \{x \in \mathbb{R}_+^2 \; ; \; x_1 + x_2 \leq s_{in}\}.$$

Indeed, we assume that invasions occur with very small concentrations $x_2(0)$ so that we can consider that the solutions stay very closed to the attractive set $\mathcal{I}$ and we can reasonably approximate the dynamics by the re-

By persistently exciting, we mean that the non-negative function $D(\cdot)$ is such that $\int_0^{+\infty} D(t)dt = +\infty$, see [BD90].

duced one (5.3). From now on, we consider $D(\cdot)$ as a control variable, *i.e.*, as a measurable function of time taking values within some interval $[D_m, D_M]$ where the minimum and maximum dilution rates D_m and D_M satisfy the inequality:

$$0 < D_m < \bar{D} < \mu_1(s_{in}) < D_M. \tag{5.4}$$

Note that D_M is large enough to have the possibility to drive solutions of (5.3) to the washout of both species. To introduce resilience, we consider a threshold

$$x_1^r \in (\bar{x}_1, s_{in}),$$

for species 1 aiming at keeping x_1 above x_1^r as much as possible. It is then natural to introduce the subset of $\mathcal{S}$, $\mathcal{K}(x_1^r)$, defined as:

$$\mathcal{K}(x_1^r) := \{x \in \mathcal{S}\ ;\ x_1 \geq x_1^r \ \text{ and } \ x_2 > 0\},$$

and to ask about weak invariance properties of $\mathcal{K}(x_1^r)$ for the dynamics (5.3) in the context of viability theory [Aub09]. Recall that given a controlled system $\dot{x} = g(x, u)$ (with $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$) and given a closed subset $K \subset \mathbb{R}^n$, the *viability kernel*, denoted by $Viab(K)$, is defined as the largest subset of initial states $x_0 \in K$ for which there is an admissible control $u(\cdot)$ such that the unique solution $x(\cdot)$ of the dynamics associated with u and such that $x(0) = x_0$, satisfies $x(t) \in K$ for any time $t \geq 0$ (see [Aub09]). We then say that the viability kernel is *weakly invariant*. Going back to (5.3), we assume in the rest of the paper (in addition to Assumptions 2.1, 2.2) that the following assumption is fulfilled:

Assumption 2.3. *The threshold x_1^r satisfies*

$$0 < D_m < \mu_1(s_{in} - x_1^r). \tag{5.5}$$

Remark 2.1. *The choice of the three parameters D_m, D_M, and x_1^r is crucial throughout this work. In this approach, note that we first chose D_m, D_M satisfying* (5.4)*, and then we suppose that the threshold x_1^r satisfies* (5.5)*. It is worth to mention that we could alternatively fix $x_1^r \in (\bar{x}_1, s_{in})$ and then choose the minimal and maximal dilution rates in such a way that* (5.5) *and the inequality $\mu_1(s_{in}) < D_M$ are*

This means that for D_M sufficiently large ($D_M > \mu_1(s_{in})$), solutions of (5.3) converge to the origin.

verified.

One has the following property, in terms of viability analysis (hereafter $cl\ S$ is the closure of a set $S \subset \mathbb{R}^2$).

Lemma 2.1. *The viability kernel $Viab(cl\,\mathcal{K}(x_1^r))$ of the set $\mathcal{K}(x_1^r)$ for* (5.3) *satisfies*

$$Viab(cl\,\mathcal{K}(x_1^r)) = [x_1^r, s_{in}] \times \{0\}.$$

Proof. Recall that x_1^r is such that $x_1^r \in (\bar{x}_1, s_{in} - \lambda_1(D_m))$ (by (5.5)). Assumptions 2.1, 2.2 and (5.4) also imply the inequality $0 < \lambda_2(D_m) < \lambda_1(D_m) < \bar{s}$. Take an initial condition $(x_{1,0}, x_{2,0})$ in $cl\,\mathcal{K}(x_1^r)$. If $x_{2,0} = 0$, the solution of (5.3) verifies $x_2(t) = 0$ for any $t > 0$ and any time-varying $D(\cdot)$. From (5.5), we can choose a constant D such that $0 < D_m \leq D < \mu_1(s_{in} - x_1^r)$, and we observe that (5.3) with $D(t) = D$ satisfies

$$x_1 = x_1^r \Rightarrow \dot{x}_1 = (\mu_1(s_{in} - x_1^r) - D)x_1^r > 0 \quad \text{and} \quad x_1 = s_{in} \Rightarrow \dot{x}_1 = -Ds_{in} < 0.$$

Thus, this constant D prevents $x_1(\cdot)$ to leave the interval $[x_1^r, s_{in}]$ over $[0, +\infty)$. Assume now that $x_{2,0} > 0$. Then, a solution $x(\cdot)$ of (5.3) associated with an admissible time-varying function $D(\cdot)$ verifies $x_2(t) > 0$ for any time $t \geq 0$. Suppose by contradiction that $x(\cdot)$ stays in $\mathcal{K}(x_1^r)$ for any time $t \geq 0$. Then one has $s(t) < s_{in} - x_1^r < \bar{s}$ for any time $t \geq 0$. By Assumption 2.2, one has for any $t \geq 0$, $\mu_1(s(t)) - \mu_2(s(t)) < 0$ and thus, we deduce the inequality

$$\begin{aligned} \dot{s}(t) &> -\mu_2(s(t))x_1(t) - \mu_2(s(t))x_2(t) + D_m(s_{in} - s(t)) \\ &= (D_m - \mu_2(s(t)))(s_{in} - s(t)), \quad t \geq 0. \end{aligned}$$

Notice that any positive solution $\zeta(\cdot)$ of $\dot{\zeta} = (D_m - \mu_2(\zeta(t)))(s_{in} - \zeta(t))$ converges to $\lambda_2(D_m)$ when $t \to +\infty$. From (5.5) one has $\lambda_2(D_m) < s_{in} - x_1^r$, hence there exist $t_1 \geq 0$ and $\underline{s} \in (0, \lambda_2(D_m))$ such that one has $\zeta(t) > \underline{s}$ for any $t \geq t_1$. From the preceding inequality, we deduce that $s(\cdot)$ satisfies $s(t) \geq \zeta(t)$ for any time $t \geq 0$. We thus deduce that for any time $t \geq t_1$, one has

$$\mu_1(s(t)) - \mu_2(s(t)) \leq c := \min\{\mu_1(\sigma) - \mu_2(\sigma)\ ;\ \sigma \in [\underline{s}, s_{in} - x_1^r]\} < 0.$$

If we differentiate the function $q_1 := x_1/x_2$ w.r.t. t, we find that

$$\dot{q}_1 = \Big(\mu_1(s(t)) - \mu_2(s(t))\Big)q_1,$$

with $s(t) = s_{in} - x_1(t) - x_2(t)$. One then obtains $\dot{q}_1 < c\, q_1$. Therefore q_1 decreases to zero and x_1 as well, leading to a contradiction. We conclude that the only solutions of (5.3) that stay in $cl\,\mathcal{K}(x_1^r)$ for any time are the ones starting with $x_{2,0} = 0$ as was to be proved. □

This lemma shows that for any given threshold x_1^r satisfying (5.5), *i.e.*, such that

$$\bar{x}_1 < x_1^r < s_{in} - \lambda_1(D_m),$$

(or equivalently $\lambda_1(D_m) < s_{in} - x_1^r < \bar{s}$), the dynamics (5.3) is not resilient for the domain $\mathcal{K}(x_1^r)$ in presence of species 2. This is precisely our starting point to introduce the concept of weak resilience.

Definition 2.1. *Let $x_1^r \in (\bar{x}_1, s_{in} - \lambda_1(D_m))$. The system* (5.3) *is said to be weakly resilient for the set $\mathcal{K}(x_1^r)$ if for any initial condition in $\mathcal{K}(x_1^r)$, there exists a time-varying function $D(\cdot)$ with values in $[D_m, D_M]$ such that the corresponding solution of* (5.3) *satisfies*

$$\text{meas}\,\{t \geq 0\ ;\ x(t) \in \mathcal{K}(x_1^r)\} = +\infty.$$

Such a function $D(\cdot)$ will be called a weakly resilient removal rate.

This definition is related to the *minimal time crisis* of controlled dynamics, studied in [BR16, BR17, BR19, DSP97], although we do not look in this paper for control functions minimizing the time spent outside the set $\mathcal{K}(x_1^r)$ over a given time period $[0, T]$.

3 Construction of a weakly resilient removal rate

The aim of this section is to propose a robust and systematic way to build a time-varying $D(\cdot)$ taking alternatively the values D_m and D_M, and that is weakly resilient for (5.3), without requiring a precise knowledge of the expressions of the growth functions $\mu_i(\cdot)$. Recall that we suppose Assumptions 2.1, 2.2, and 2.3 to be fulfilled and that the parameter x_1^r satisfies

$$x_1^r \in (\bar{x}_1, s_{in} - \lambda_1(D_m)).$$

We begin by giving the main result (in Section 3.1) which gives a construc-

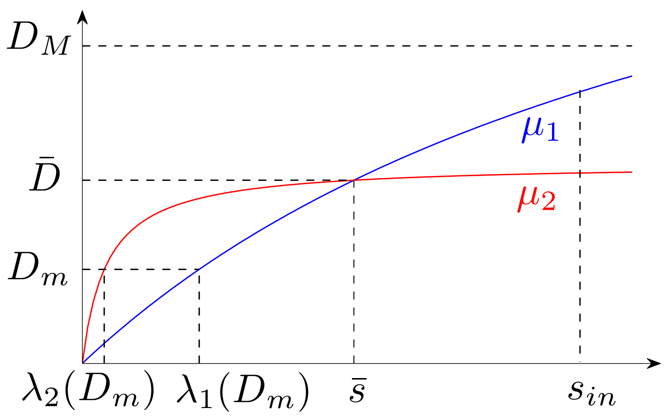

FIGURE 5.1 – *Illustration of Assumption 2.3.*

tion of a weakly resilient $D(\cdot)$. Next, we provide some properties of the dynamical system(5.3) in the domain $\mathcal{S}$ for a constant D ($D = D_M$ and $D = D_m$). Finally, we give the proof of Proposition 3.1 at the end of this section.

3.1 Synthesis of a weakly resilient removal rate

In the following Proposition, we propose a time-varying $D(\cdot)$ allowing the dynamics (5.3) to be weakly resilient for the set $\mathcal{K}(x_1^r)$. For any $\varepsilon > 0$, we define the set $\mathcal{E}$

$$\mathcal{E} := (0, \bar{x}_1] \times (0, \varepsilon].$$

Proposition 3.1. *For ε small enough and any initial condition $x_0 := (x_{1,0}, x_{2,0}) \in \mathcal{K}(x_1^r)$ there exists a piecewise constant function $D(\cdot)$ which alternates the values D_M, D_m on time intervals $[T_i, T_{i+1})$, $i \in \mathbb{N}$ satisfying:*

$$T_0 = 0 < T_1 < \cdots < T_i < T_{i+1} < \cdots \quad \text{and} \quad \lim_{i \to \infty} T_i = +\infty, \tag{5.6}$$

where $x(\cdot)$ is the unique solution of (5.3) *associated with the time-varying $D(\cdot)$ such that*

(i) *if $x(T_i) \notin \mathcal{E}$, one has $D(t) = D_M$ for $t \in [T_i, T_{i+1})$ with T_{i+1} is defined as the first next entry time in $\mathcal{E}$.*

(ii) *if $x(T_i) \in \mathcal{E}$, one has $D(t) = D_m$ for $t \in [T_i, T_{i+1})$, the trajectory $x(\cdot)$ enters to the set $\mathcal{K}(x_1^r)$ in finite time and T_{i+1} is defined as the first next exit time from $\mathcal{K}(x_1^r)$.*

Finally, the time-varying $D(\cdot)$ is a weakly resilient removal rate.

We begin by a lemma which describes the asymptotic behavior of (5.3) when D is constant.

Lemma 3.1. *Any solution of* (5.3) *in $\mathcal{S}$ with a constant removal rate D converges asymptotically to an equilibrium.*

Proof. First, consider an initial condition on the axes that are invariant by (5.3). Then, the variable x_i (i equal to 1 or 2) is solution of a scalar autonomous dynamics on the x_i-axis. Therefore, either it converges to an equilibrium point on the axis, or it tends to infinity, which is not possible as the domain $\mathcal{S}$ is bounded.

Consider now a positive initial condition in the set $\mathcal{S}$. The corresponding solution then remains in the positive orthant, and one can consider the variables $\xi_i = \ln(x_i)$, $i = 1, 2$, whose dynamics is

$$\dot{\xi}_i = F_i(\xi) := \mu_i\left(s_{in} - e^{\xi_1} - e^{\xi_2}\right) - D, \quad i = 1, 2.$$

For a constant D, one has

$$\mathrm{div}F(\xi) = \sum_{i=1,2} \partial_{\xi_i} F_i(\xi) = -\sum_{i=1,2} \mu_i\left(s_{in} - e^{\xi_1} - e^{\xi_2}\right) e^{\xi_i} < 0.$$

By Dulac's criterion, the system has no closed orbit and by Poincaré-Bendixon Theorem (see [Per13]). We can then conclude that solutions of (5.3) converge asymptotically to an equilibrium, since trajectories are bounded. □

In the sequel, we denote by $z(\cdot, \zeta, D)$ the unique solution of (5.3) (over $\mathbb{R}$) for an initial condition $z(0) = \zeta \in \mathcal{S}$ and a constant $D \in \{D_m, D_M\}$.

3.2 Properties of the reduced dynamics with constant $D = D_M$

We now provide asymptotic properties of (5.3) when D is constant equal to D_M, that are illustrated on Fig. 5.2.

Lemma 3.2. *Any solution $z(\cdot, \zeta, D_M)$ of* (5.3) *with $\zeta \in \mathcal{S}$ converges asymptotically to the origin. Moreover, for any positive initial condition ζ in $\mathcal{S}$, $z_1(\cdot, \zeta, D_M)$ and $z_2(\cdot, \zeta, D_M)$ are decreasing functions and the trajectory converges to the origin tangentially to the x_1-axis.*

Proof. From Assumption 2.3, the system (5.3) has a unique steady state $(0, 0)$ in $\mathcal{S}$. From Lemma 3.1, we deduce that it is globally asymptotically stable on $\mathcal{S}$. Consider a positive initial condition. From the expression of the dynamics (5.3), the solution $x(\cdot) = z(\cdot, \zeta, D_M)$ is clearly positive for any $t \geq 0$, and by Assumption 2.3, one gets $\dot{x}_i(t) < 0$ for any $t \geq 0$, thus $z_i(\cdot, \zeta, D_M)$ is decreasing for $i = 1, 2$. Consider then the function $q_2 := x_2/x_1$. A straightforward computation of its derivative yields

$$\dot{q}_2 = \Big(\mu_2(s_{in} - x_1(t) - x_2(t)) - \mu_1(s_{in} - x_1(t) - x_2(t))\Big)q_2.$$

By Assumption 2.2, one has $\mu_2(s_{in}) - \mu_1(s_{in}) < 0$. Since $x(\cdot)$ converges to 0, we deduce that there exist $\eta > 0$ and $t_M > 0$ such that $\dot{q}_2(t) < -\eta q_2(t)$ for any time $t > t_M$. This proves that $q_2(\cdot)$ converges asymptotically to 0 and that trajectories are tangent to the x_1-axis at $(0, 0)$. □

We are now in a position to introduce the following notation that will be used hereafter (see Fig. 5.2):

- Consider the point $\hat{P}^r := (x_1^r, s_{in} - x_1^r)$ on the boundary of $\mathcal{S}$ and set

$$\hat{x}(\cdot) := z(\cdot, \hat{P}^r, D_M).$$

- The forward semi-orbit of (5.3) with $D = D_M$ passing through P^r is denoted by (see Fig. 5.2):

$$\hat{\gamma}^+ := \{\hat{x}(t)\,;\, t \geq 0\}.$$

- In view of Lemma 3.2, $\hat{x}_1(\cdot)$ is decreasing and thus reaches $\bar{x}_1$ in finite time. Hence, there are $\hat{t} > 0$ and $\delta > 0$ satisfying:

$$\hat{t} := \inf\{t > 0;\ \hat{x}_1(t) < \bar{x}_1\} \quad \text{and} \quad \delta := \hat{x}_2(\hat{t}).$$

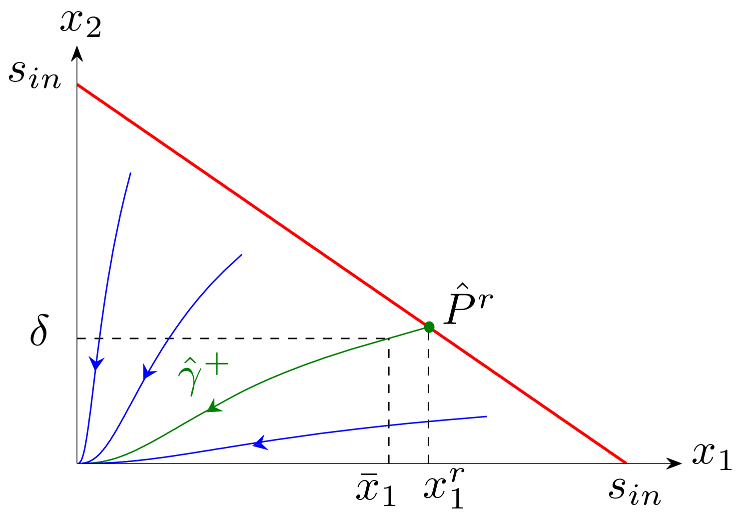

FIGURE 5.2 – *Phase portrait of* (5.3) *with constant* $D = D_M$ *and plot of the points* $\hat{P}^r$*,* $(\bar{x}_1, \delta)$*, and the semi-orbit* $\hat{\gamma}^+$.

3.3 Properties of the reduced dynamics with constant $D = D_m$

We now turn to asymptotic properties of (5.3) with $D = D_m$. In the sequel, we shall use the notation

$$x_i^m := s_{in} - \lambda_i(D_m), \quad i = 1, 2.$$

The scalar product in $\mathbb{R}^2$ is written $a \cdot b$ with $a, b \in \mathbb{R}^2$ and $\|a\|$ denotes the euclidean norm of a vector $a \in \mathbb{R}^2$.

Lemma 3.3. *The system* (5.3) *with* $D = D_m$ *possesses the following properties.*

(i) *It admits exactly three equilibria in* $\mathcal{S}$: $E_0 := (0,0)$, $E_1^m := (x_1^m, 0)$ *and* $E_2^m := (0, x_2^m)$ *which are respectively an unstable node, a saddle point, and a stable node.*

(ii) *One has* $x_1^m < x_2^m$ *and the "strip"* $\mathcal{S}_{1,2}^m$ *defined as*

$$\mathcal{S}_{1,2}^m := \{x \in \mathcal{S} \; ; \; x_1 + x_2 \in [x_1^m, x_2^m]\} \, ,$$

is invariant (by (5.3) *with* $D = D_m$*).*

(iii) *The edge* $(0, s_{in}] \times \{0\}$ *is the stable manifold of* (5.3) *with* $D = D_m$ *at* E_1^m *on* $\mathcal{S}$*. The unstable manifold at* E_1^m *in the domain* $\mathcal{S}$ *is denoted by* $W^u(E_1^m)$*: it connects* E_1^m *to* E_2^m *and satisfies* $W^u(E_1^m) \subset \mathcal{S}_{1,2}^m$*.*

Proof. Inequality (5.5) and the monotonicity of the functions $\mu_i(\cdot)$ (Assumption 2.1) imply that there are two equilibria of (5.3) in $\mathcal{S}$ that are distinct of the origin and on the axes. They are uniquely defined by E_1^m and E_2^m. The Jacobian matrix at E_1^m and E_2^m are respectively given by

$$J(E_1^m) := \begin{bmatrix} -\mu_1'(s_{in} - x_1^m)x_1^m & -\mu_1'(s_{in} - x_1^m)x_1^m \\ 0 & \mu_2(s_{in} - x_1^m) - D_m \end{bmatrix},$$

$$J(E_2^m) := \begin{bmatrix} \mu_1(s_{in} - x_2^m) - D_m & 0 \\ -\mu_2'(s_{in} - x_2^m)x_2^m & -\mu_2'(s_{in} - x_2^m)x_2^m \end{bmatrix}.$$

The point E_1^m is a saddle point because the eigenvalues of $J(E_1^m)$ are of opposite sign, E_2^m is a stable node because the eigenvalues of $J(E_2^m)$ are negative and E_0 is clearly an unstable node which proves (i). It is worth noting that the unstable manifold $W^u(E_1^m)$ necessarily connects E_1^m to E_2^m by Lemma 3.1.

Let us now prove (ii). From Assumption 2.2, one has $x_1^m < x_2^m$. When $x \in \mathcal{S}_{1,2}^m$ is such that $x_1 + x_2 = x_1^m$, one has $\dot{x}_1 = 0$ and $\dot{x}_2 > 0$ whereas if $x_1 + x_2 = x_2^m$, one has $\dot{x}_1 < 0$ and $\dot{x}_2 = 0$. Hence $\mathcal{S}_{1,2}^m$ is invariant.

Let us finally prove (iii). The positive half axis $x_1 > 0$ is clearly the stable manifold of (5.3) with $D = D_m$ at E_1^m. Consider a non-null eigenvector v^+ of $J(E_1^m)$ associated with the positive eigenvalue $\mu_2(s_{in} - x_1^m) - D_m$, and let

n be an outward normal to $\mathcal{S}_{1,2}^m$ at E_1^m. A straightforward calculation yields

$$v^+ = \begin{bmatrix} -1 \\ 1 + \frac{\mu_2(\lambda_1(D_m))-D_m}{x_1^m \mu_1'(\lambda_1(D_m))} \end{bmatrix}, \quad n = \begin{bmatrix} -1 \\ -1 \end{bmatrix},$$

implying that

$$v^+ \cdot n = -\frac{\mu_2(\lambda_1(D_m)) - D_m}{x_1^m \mu_1'(\lambda_1(D_m))} < 0.$$

One then concludes that the vector v^+ points inward $\mathcal{S}_{1,2}^m$ at E_1^m. On another hand, from the Theorem of the stable and unstable manifolds [Per13], we know that $W^u(E_1^m)$ is tangent to v^+ at E_1^m. Therefore, there is a neighborhood $\mathcal{V}$ of E_1^m in $\mathcal{S}_{1,2}^m$ such that $W^u(E_1^m) \cap \mathcal{V} \subset \mathcal{S}_{1,2}^m$, but, as $\mathcal{S}_{1,2}^m$ is invariant, we conclude that $W^u(E_1^m) \subset \mathcal{S}_{1,2}^m$ as was to be proved. □

Recall that the unstable manifold $W^u(E_1^m)$ is a trajectory of (5.3) with $D = D_m$, and that (5.3) satisfies $\dot{x}_1 < 0$ on $int\ \mathcal{S}_{1,2}^m$. So, $W^u(E_1^m)$ can be parametrized as a function $x_1 \mapsto w_u(x_1)$, $x_1 \in [0, x_1^m]$. Hereafter, $\mathrm{hyp}(w_u)$ stands for the hypograph of w_u and let $\mathcal{D} \subset \mathcal{S}$ be defined as (see Fig. 5.3):

$$\mathcal{D} := \mathrm{hyp}(w_u) \cap \mathcal{S}.$$

Note that the domain $\mathcal{D}$ is necessarily forward and backward invariant (for (5.3) with $D = D_m$) as its boundary is a locus of trajectories. Similarly as with $D = D_M$, let us introduce the following notation (see Fig. 5.3):

- Since there is a unique intersection point between $W^u(E_1^m)$ and the line $\{x_1 = x_1^r\}$, we set:
$$\bar{x}_2 = w_u(x_1^r), \tag{5.7}$$
one can also write $(x_1^r, \bar{x}_2) = W^u(E_1^m) \cap \{x_1 = x_1^r\}$.
- From Assumption 2.3, one has $x_1^r < x_1^m$, thus we can fix a point $\check{P}^r := (x_1^r, \check{x}_{2,0}) \in \mathcal{D}$ such that:
$$0 < \check{x}_{2,0} < \min(\delta, x_1^m - x_1^r), \tag{5.8}$$
where $\check{x}_{2,0}$ is small enough to ensure $\check{P}^r \in \mathcal{D}$.
- The backward semi-orbit of (5.3) with $D = D_m$ passing though $\check{P}^r$ is

denoted by:

$$\check{\gamma}^- := \{\check{x}(t)\,;\, t \leq 0\},$$

where $\check{x}(\cdot) = z(\cdot, \check{P}^r, D_m)$.

- Finally, define a positive parameter $\eta > 0$ as $\eta := x_1^m - x_1^r - \check{x}_{2,0}$.

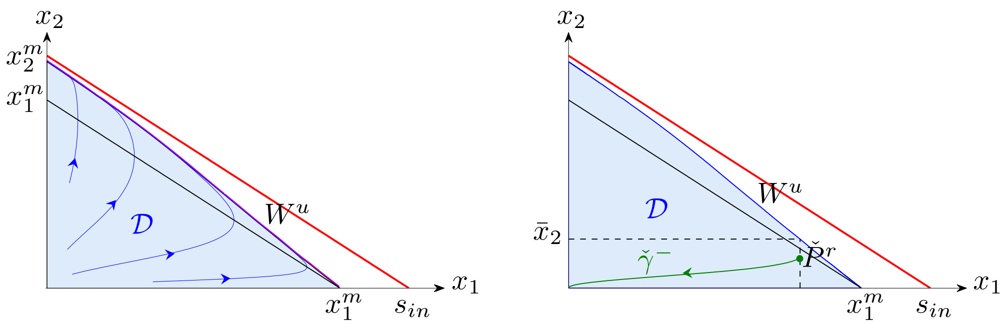

FIGURE 5.3 – *Left: phase portrait of* (5.3) *with constant* $D = D_m$ *in the domain* $\mathcal{D}$ *whose boundary is the unstable manifold* $W^u(E_1^m)$*. Right: plot of the point* $\check{P}^r$ *and the semi-orbit* $\check{\gamma}^-$*.*

Remark 3.1. (i) *From Lemma 3.3* (i)*, any solution of* (5.3) *with* $D = D_m$ *in* $\mathcal{D} \setminus ([0, x_1^m] \times \{0\})$ *converges to* E_2^m*. Note also that for* $(x_1, x_2) \in \mathcal{D} \setminus \{E_2^m\}$*, one has* $x_1 + x_2 < x_2^m$*, which implies the inequality* $\mu_2(s_{in} - x_1 - x_2) > D_m$*. Hence, any solution* $x(\cdot)$ *in* $\mathcal{D} \setminus \{E_2^m\}$ *satisfies* $\dot{x}_2 > 0$*. Moreover, one has:*

$$(x_1, x_2) \in \{x \in \mathcal{D}\,;\, x_1 > 0 \text{ and } x_1 + x_2 < x_1^m\} \;\Rightarrow\; \dot{x}_1 > 0,$$
$$(x_1, x_2) \in \{x \in \mathcal{D}\,;\, x_1 > 0 \text{ and } x_1 + x_2 > x_1^m\} \;\Rightarrow\; \dot{x}_1 < 0.$$

(ii) *As* $W^u(E_1^m)$ *is a trajectory, any solution of* (5.3) *with* $D = D_m$ *crosses the line* $\{x_1 = x_1^r\}$ *at some point* (x_1^r, x_2) *such that* $x_2 < \bar{x}_2$*.*
(iii) *Since* $\mathcal{D}$ *is backward invariant by* (5.3) *with* $D = D_m$ *and* $\check{P}^r \in \mathcal{D}$*, the inclusion* $\check{\gamma}^- \subset \mathcal{D}$ *is fulfilled.*

The next lemma will be useful to define a small $\varepsilon > 0$ and times $\hat{T}$ and $\check{T}$ (see Remarks 3.2 and 3.3 below).

Lemma 3.4. *The curves* $\hat{\gamma}^+$ *and* $\check{\gamma}^-$ *intersect in the domain* $(0, \bar{x}_1) \times (0, \check{x}_{2,0})$.

Proof. Consider the variable $\check{q}_1 = \check{x}_1/\check{x}_2$ on the positive orthant. As previously, one has

$$\dot{\check{q}}_1 = \Big(\mu_1(s_{in} - \check{x}_1(t) - \check{x}_2(t)) - \mu_2(s_{in} - \check{x}_1(t) - \check{x}_2(t))\Big)\check{q}_1.$$

From Assumption 2.2, $\mu_1(s_{in}) - \mu_2(s_{in}) > 0$. As $\check{x}(t) \to (0,0)$ when $t \to -\infty$, there exist $\check{t} < 0$ and $c' > 0$ such that

$$\mu_1(s_{in} - \check{x}_1(t) - \check{x}_2(t)) - \mu_2(s_{in} - \check{x}_1(t) - \check{x}_2(t)) > c', \quad t \le \check{t},$$

which shows that $\check{q}_1(t)$ tends to 0 when t tends to $-\infty$. Therefore $\check{\gamma}^-$ is tangent to the x_2-axis at E_0. As $\hat{\gamma}^+$ is tangent to the x_1-axis at E_0 (Lemma 3.2), we deduce that the curve $\check{\gamma}^-$ is above the curve $\hat{\gamma}^+$, in a neighborhood of the point E_0 in $\mathcal{S}$. Since $\check{x}_1(0) = x_1^r > \bar{x}_1$ and $\check{x}_1(t)$ tends to 0 when t tends to $-\infty$, there exists $\bar{t} < 0$ such that

$$\check{x}_1(\bar{t}) = \bar{x}_1 \quad \text{with} \quad \bar{t} = \sup\{t < 0;\ \check{x}_1(t) = \bar{x}_1\}.$$

This means that $\check{x}_1(t) > \bar{x}_1$ for all $t \in (\bar{t}, 0]$. As $\check{x}_2(\cdot)$ is increasing on $\mathcal{D}$ (*cf.* Remark 3.1), one has $\check{x}_2(\bar{t}) < \check{x}_2(0)$ and from the choice of $\check{x}_{2,0} = \check{x}_2(0) < \delta$ (see condition (5.8)), one gets $\check{x}_2(t) < \delta$, for all $t \le 0$ and in particular at $t = \bar{t}$. The point $(\bar{x}_1, \check{x}_2(\bar{t}))$ of $\check{\gamma}^-$ is thus below $(\bar{x}_1, \delta)$, which belongs to $\hat{\gamma}^+$. Therefore, $\hat{\gamma}^+$ and $\check{\gamma}^-$ have to cross at some point $\check{x}(\check{T})$ with $\check{T} < \bar{t}$, which verifies $0 < \check{x}_1(\check{T}) < \bar{x}_1$ and $0 < \check{x}_2(\check{T}) < \check{x}_{2,0}$. □

Remark 3.2. *Since $\check{\gamma}^- \subset \mathcal{D}$ (cf. Remark 3.1), the intersection between $\check{\gamma}^-$ and $\hat{\gamma}^+$ is also contained in $\mathcal{D}$.*

Lemma 3.4 implies that for each choice of the point $\check{P}^r$, there is an intersection point

$$P_\varepsilon := (\underline{x}_1, \varepsilon) \in \mathcal{D},$$

between $\hat{\gamma}^+$ and $\check{\gamma}^-$ such that

$$0 < \underline{x}_1 < \bar{x}_1 \quad \text{and} \quad 0 < \varepsilon < \check{x}_{2,0},$$

see Fig. 5.4. By construction, there are $\hat{T} > 0$ and $\check{T} > 0$ such that $P_\varepsilon = \hat{x}(\hat{T}) = \check{x}(-\check{T})$. Recall that $\mathcal{E}$ is by definition

$$\mathcal{E} := (0, \bar{x}_1] \times (0, \varepsilon].$$

Since $\bar{x}_1 < x_1^r$ and $\varepsilon < \check{x}_{2,0}$, the corner point $(\bar{x}_1, \varepsilon)$ of $\mathcal{E}$ is below the curve $\check{\gamma}^-$. Thus, one has also the inclusion $\mathcal{E} \subset \mathcal{D}$ (because $\check{\gamma}^- \subset \mathcal{D}$), see Fig. 5.4.

Remark 3.3. *Given D_m, D_M and x_1^r that fulfill Assumption 2.3, the parameter ε can be chosen arbitrarily small taking the parameter $\check{x}_{2,0}$ in the definition of $\check{P}^r$ small enough (recall* (5.8)*).*

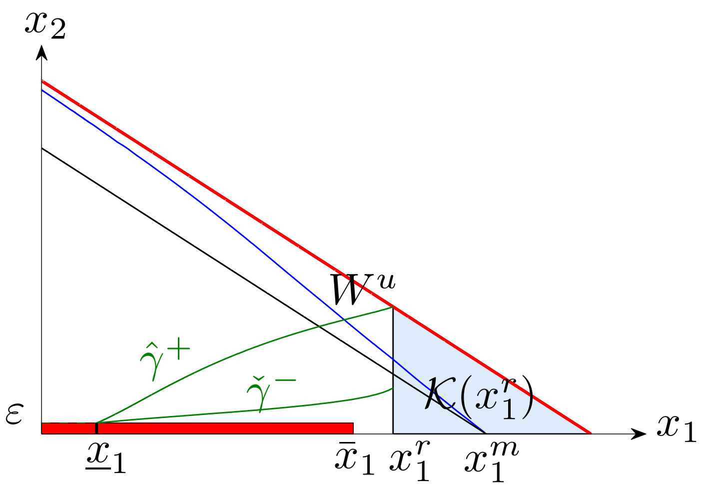

FIGURE 5.4 – *Plot of the set $\mathcal{E} := (0, \bar{x}_1] \times (0, \varepsilon]$ (in red) and the intersection point $P_\varepsilon = (\bar{x}_1, \varepsilon)$ between $\hat{\gamma}^+$ and $\check{\gamma}^-$. The set $\mathcal{K}(x_1^r)$ is depicted in blue.*

3.4 Proof of Proposition 3.1

To help the reader, we provide in Appendix 1 a list of the notations used in Section 3.

We start by giving the following definition.

Definition 3.1. *The "southeast" order in $\mathbb{R}^2$ (denoted by $\preccurlyeq$) is defined as*

$$\forall (x, y) \in \mathbb{R}^2, \quad x \preccurlyeq y \Longleftrightarrow \{x_1 \leq y_1,\ x_2 \geq y_2\}.$$

Notice that the dynamics (5.3) is competitive, and therefore (5.3) preserves the order $\preccurlyeq$ (see [Smi08]): for any admissible time-varying function $D(\cdot)$, one has:

$$\forall (\zeta_1, \zeta_2) \in \mathcal{S}, \quad \zeta_1 \preccurlyeq \zeta_2 \Longrightarrow z(t, \zeta_1, D(t)) \preccurlyeq z(t, \zeta_2, D(t)), \quad \forall t \geq 0. \tag{5.9}$$

We have now given the necessary definitions and properties so that we can give to the proof of Proposition 3.1.

Proof. For clarity, we present the proof in several steps.

Step 1. Fix an initial condition $x_0 \in \mathcal{K}(x_1^r)$ with $x_{2,0} < s_{in} - x_1^r$. As $x_0 \notin \mathcal{E}$, we set $D = D_M$. Accordingly to Lemma 3.2, $x(\cdot)$ converges asymptotically to E_0 and thus $T_1 := \inf\{t > 0 \ ; \ x(t) \in \mathcal{E}\}$ is well defined. Notice that there are two ways to reach $\mathcal{E}$:

(i) $x_1(T_1) = \bar{x}_1$ and $x_2(T_1) < \varepsilon$,

(ii) $x_1(T_1) \leq \bar{x}_1$ and $x_2(T_1) = \varepsilon$.

Let us show that in both cases one has $x_1(T_1) > \underline{x}_1$. As $\bar{x}_1 > \underline{x}_1$ (recall Lemma 3.4), we get $x_1(T_1) > \underline{x}_1$ in case (i). In case (ii), one can also write $T_1 = \inf\{t > 0 \ ; \ z_2(t, x_0, D_M) \leq \varepsilon\}$. The order property (5.9) then implies that

$$z(t, \hat{P}^r, D_M) \preccurlyeq z(t, x_0, D_M), \quad t \geq 0.$$

As $z(\hat{T}, \hat{P}^r, D_M) = \hat{x}(\hat{T}) = (\underline{x}_1, \varepsilon)$, we deduce that

$$z_1(\hat{T}, x_0, D_M) \geq \underline{x}_1, \quad z_2(\hat{T}, x_0, D_M) \leq \varepsilon.$$

From the definition of T_1, one has $T_1 < \hat{T}$ (one gets $T_1 = \hat{T}$ if $x_0 = \hat{P}^r$) and thus $x_1(T_1) = z_1(T_1, x_0, D_M) \geq \underline{x}_1$. If $x_1(T_1) = \underline{x}_1$, one should then have $x_2(T_1) = \varepsilon$, that is, $z(\cdot, x_0, D_M) = z(\cdot, \hat{P}^r, D_M)$ which is not true. Hence, we obtain as well $x_1(T_1) > \underline{x}_1$ as was to be proved. In addition, notice that $P_\varepsilon \preccurlyeq x(T_1)$. Since $\mathcal{E} \subset \mathcal{D}$, the point $x(T_1)$ necessarily satisfies $x(T_1) \in \mathcal{D}$.

Step 2. At $t = T_1$, we set $D = D_m$. We use again the order property (5.9):

$$P_\varepsilon \preccurlyeq x(T_1) \Longrightarrow z(t, P_\varepsilon, D_m) \preccurlyeq z(t, x(T_1), D_m), \quad \forall t \geq 0, \tag{5.10}$$

and, as we have shown that $x_1(T_1) > \underline{x}_1$, we obtain the inequalities

$$z_1(\check{T}, x(T_1), D_m) > z_1(\check{T}, P_\varepsilon, D_m) = x_1^r, \quad z_2(\check{T}, x(T_1), D_m) \leq z_2(\check{T}, P_\varepsilon, D_m) = \check{x}_{2,0}.$$

Therefore one has $x(T_1 + \check{T}) = z(\check{T}, x(T_1), D_m) \in int\ \mathcal{K}(x_1^r)$. One can then define a time $\bar{T}_1$ as:

$$\bar{T}_1 := \inf\{t > T_1 \ ; \ x_1(t) > x_1^r\},$$

which is such that $\bar{T}_1 \in (T_1, T_1 + \check{T})$. From the monotonicity of $x_2(\cdot)$ in the set $\mathcal{D}$ (*cf.* Remark 3.1), one obtains

$$x_2(\bar{T}_1) < x_2(T_1 + \check{T}) = z_2(\check{T}, x(T_1), D_m) \leq \check{x}_{2,0}.$$

As $x(T_1)$ belongs to the set $\mathcal{D}$ with $x_1(T_1) > 0$, $z(\cdot, x(T_1), D_m)$ converges asymptotically to the equilibrium E_2^m that lies on the x_2-axis (*cf.* Remark 3.1). The time $T_2 > T_1 + \check{T}$ such that $x_1(T_2) = x_1^r$, where $x(t) := z(t - T_1, x(T_1), D_m)$, $t \geq T_1$, is thus well defined. Moreover one has $x(T_2) \in \mathcal{K}(x_1^r)$ and $x(T_2) \in \mathcal{D}$ as $\mathcal{D}$ is invariant by (5.3) (for $D = D_m$).

Step 3. At time T_2, we have shown that $x(T_2)$ belongs to the set $\mathcal{K}(x_1^r)$, and also to the set $\mathcal{D}$ which implies that $x_2(T_2) < s_{in} - x_1^r$. Therefore, we can consider $x(T_2)$ as a new initial condition and apply iteratively the results of steps 1 and 2, defining an increasing sequence of times $(T_i)_{i \in \mathbb{N}}$. For $i = 2k+1$ (with $k \in \mathbb{N}$), one has $x(T_{2k+1}) \in \mathcal{E}$ and, as in step 2, we can define $\bar{T}_{2k+1} \in (T_{2k+1}, T_{2k+1} + \check{T})$ such that

$$x_1(\bar{T}_{2k+1}) = x_1^r \quad \text{with} \quad \bar{T}_{2k+1} := \inf\{t > T_{2k+1} \; ; \; x_1(t) > x_1^r\}. \tag{5.11}$$

As shown in step 2, we necessarily have

$$x_2(\bar{T}_{2k+1}) < \check{x}_{2,0}. \tag{5.12}$$

Because $T_{2k+2} > T_{2k+1} + \check{T}$, we get that $\lim_{i \to +\infty} T_i = +\infty$ which concludes that property (5.6) is fulfilled.
Note that if we chose $x_0 = \hat{P}^r$ then one obtains $T_1 = \hat{T}$ (and then by the unicity of the solution $x(T_1) = P_\varepsilon$). Furthermore, in this case, the time $\bar{T}_1$ is such that $\bar{T}_1 = T_1 + \check{T}$ with $x(\bar{T}) \in \mathcal{K}(x_1^r)$, since $x(T_1) = \check{x}(-\check{T})$. The time $T_2 > \bar{T}_1$ that is the first exit time from $\mathcal{K}(x_1^r)$ is well defined with $x_2(T_2) < s_{in} - x_1^r$.

Step 4. We now show that the time spent by $x_1(\cdot)$ above the threshold x_1^r is of infinite measure. For each $k \in \mathbb{N}$, one has, from the definition of T_{2k+2}:

$$\text{meas}\{t \in [T_{2k+1}, T_{2k+2}] \; ; \; x_1(t) > x_1^r\} = T_{2k+2} - \bar{T}_{2k+1}.$$

From Remark 3.1, one has $\dot{x}_1(t) > 0$ when $s(t) = s_{in} - x_1(t) - x_2(t) > \lambda_1(D_m)$.

At time $\bar{T}_{2k+1}$, the inequality (5.12) implies that $s(\bar{T}_{2k+1}) > s_{in} - x_1^r - \check{x}_{2,0}$. We deduce that

$$s(\bar{T}_{2k+1}) - \lambda_1(D_m) = s(\bar{T}_{2k+1}) - (s_{in} - x_1^r - \check{x}_{2,0}) + \eta > \eta > 0. \tag{5.13}$$

Next, let us define

$$\tau_k := \sup\{\theta > 0 \ ; \ s(\bar{T}_{2k+1} + \theta) > \lambda_1(D_m)\}.$$

Then, one has necessarily $x_1(t) > x_1^r$ for any $t \in [\bar{T}_{2k+1}, \bar{T}_{2k+1} + \tau_k]$ and one obtains the inequality

$$T_{2k+2} - \bar{T}_{2k+1} > \tau_k.$$

Let us now give a lower bound on the value of τ_k. From equations (5.3), the following properties hold true:

$$\left\{\begin{array}{lcl} x_1 & > & x_1^r, \\ s & > & \lambda_1(D_m) \end{array}\right. \Rightarrow \left\{\begin{array}{lclcl} s & = & s_{in} - x_1 - x_2 & < & s_{in} - x_1^r, \\ x_1 & = & s_{in} - s - x_2 & < & s_{in} - \lambda_1(D_m) = x_1^m, \\ x_2 & = & s_{in} - s - x_1 & < & s_{in} - \lambda_1(D_m) - x_1^r \\ & & & & = x_1^m - x_1^r, \end{array}\right.$$

from which one can obtain a lower bound on the speed at which the variable s decreases (as long as s is above $\lambda_1(D_m)$ and x_1 above x_1^r):

$$\begin{array}{rcl} \dot{s} & = & -\mu_1(s)x_1 - \mu_2(x)x_2 + D_m(s_{in} - s) \\ & \geq & -\mu_1(s_{in} - x_1^r)x_1^m - \mu_2(s_{in} - x_1^r)(x_1^m - x_1^r) := -c''. \end{array}$$

Notice that $c'' > 0$. By integrating the above inequality, one can conclude that s stays above $\lambda_1(D_m)$ for a duration larger than $(s(\bar{T}_{2k+1}) - \lambda_1(D_m))/c''$. Thanks to (5.13), we can thus write

$$\tau_k > M := \frac{\eta}{c''} > 0,$$

where $M > 0$ does not depend on k. Finally, we have shown that for each $k \in \mathbb{N}$, one has

$$\text{meas}\{t \in [0, T_{2k+2}] \ ; \ x_1(t) > x_1^r\} > kM,$$

which shows that the time-varying $D(\cdot)$ is a weaky resilient removal rate as was to be proved. □

Remark 3.4. (i) *In the proof of Proposition 3.1 (step 1), we have seen that* $x_1(T_{2k+1}) > \underline{x}_1$ *for any* $k \in \mathbb{N}$. *Therefore, Proposition 3.1 remains valid if the set* $\mathcal{E}$ *is replaced by*

$$\tilde{\mathcal{E}} := (\underline{x}_1, \bar{x}_1] \times (0, \varepsilon].$$

(ii) *Notice also that the trajectory given by Proposition 3.1 reaches the set* $\mathcal{D}$ *at some time* $t \leq T_1$, *and then remains in this set for any future time. Indeed, from Remark 3.1, the trajectory belongs to* $\mathcal{D}$ *on any time interval* $[T_{2k+1}, T_{2k+2}]$ *(with* $D = D_m$*). On a time interval* $[T_{2k+2}, T_{2k+3}]$, *we have set* $D = D_M$, *and we have seen in Lemma 3.2 that* $x_1(\cdot)$ *and* $x_2(\cdot)$ *are decreasing (with* $D = D_M$*). So, the trajectory also remains in* $\mathcal{D}$ *for* $t \in [T_{2k+2}, T_{2k+3}]$. *One then concludes that the trajectory remains in* $\mathcal{D}$ *as well as in* $\{x_1 > \underline{x}_1\}$ *(thanks to point* (i) *above) over* $[T_1, +\infty)$

(iii) *Finally, note that at times* $\bar{T}_{2k+1}$ *and* T_{2k+2}, *one has* $x_1(\bar{T}_{2k+1}) = x_1(T_{2k+2}) = x_1^r$ *and*

$$x_2(\bar{T}_{2k+1}) < x_2(T_{2k+2}) < \bar{x}_2 < x_2^m - x_1^r. \tag{5.14}$$

4 Convergence to periodic solutions

The goal of this section is to show that the weakly resilient removal rate defined by Proposition 3.1 generates asymptotically positive periodic solutions of system (5.3). Before addressing this point, we shall first prove the existence of a periodic solution of (5.3) associated with the time-varying $D(\cdot)$ given by Proposition 3.1. To this end, let us introduce an operator $\mathcal{O}$ as

$$\begin{array}{rccc} \mathcal{O}: & (0, s_{in} - x_1^r] & \to & (0, s_{in} - x_1^r] \\ & x_{2,0} & \mapsto & x_2(T_2) \end{array}$$

where $x(\cdot) = (x_1(\cdot), x_2(\cdot))$ is the unique solution of (5.3) for the initial condition (x_1^r, x_{02}) and the time-varying $D(\cdot)$ given by Proposition 3.1, parameters D_m, D_M, and $\check{P}^r$ being fixed. Notice that times T_i, $i \geq 1$, introduced in Proposition 3.1 depend on $x_{2,0}$ (in particular T_2). Hence, this operator slightly differs from the *Poincaré map* used for instance in [MS81, SW95] for finding periodic solutions of dynamical systems, for which the period is fixed beforehand. We shall next examine properties of the operator $\mathcal{O}$. Doing so, let us introduce the following notation (see Fig. 5.5):

- Denote by $\tilde{\gamma}$ and $\check{\gamma}$, the orbits of (5.3) with $D = D_m$ passing respectively by $(x_1^r, x_1^m - x_1^r)$ and $\check{P}_r$.
- Observe that $\tilde{\gamma}$ is tangent to the segment $\{x_1 = x_1^r\} \cap \mathcal{S}$ at $(x_1^r, x_1^m - x_1^r)$. Because $\check{x}(\cdot)$ converges to E_2^m (Lemma 3.3), there are exactly two intersection points between $\check{\gamma}$ and $\{x_1 = x_1^r\} \cap \mathcal{S}$, namely $\check{P}_r$ and

$$\check{Q}_r := (x_1^r, \underline{x}_2) \quad \text{with} \quad \underline{x}_2 > x_1^m - x_1^r. \tag{5.15}$$

- In the sequel, we denote by J the interval $J := [\underline{x}_2, \bar{x}_2]$ (recall (5.7)).

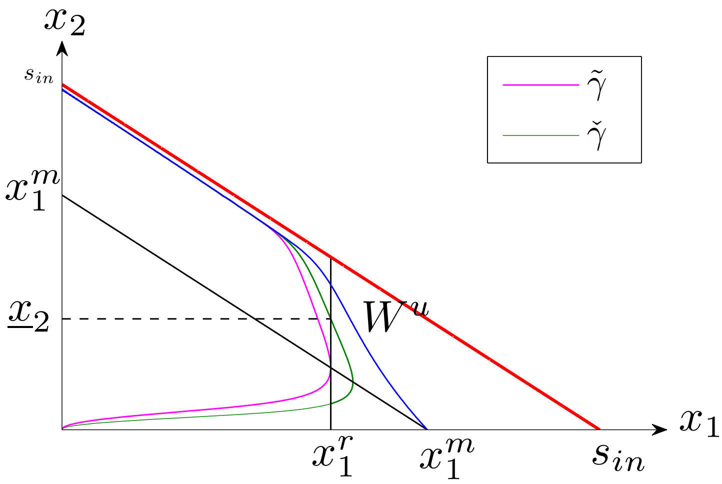

FIGURE 5.5 – *Plot of the orbits $\tilde{\gamma}$ and $\check{\gamma}$ and the point $\check{Q}^r$.*

4.1 Properties of the operator $\mathcal{O}$

In this section, we prove that $\mathcal{O}$ is continuous and decreasing. The continuity of $\mathcal{O}$ will follow from the continuity property of the first entry time into a set, that is related to Petrov's condition (see Appendix 2).

For initial condition $(x_1^r, x_{2,0})$ with $x_{2,0} \in (0, S_{in} - x_1^r]$, we denote $T_1(x_{2,0})$, $T_2(x_{2,0})$ the times T_1, T_2 given by Proposition 3.1.

Proposition 4.1. *The time-varying $D(\cdot)$ contructed in Proposition 3.1 fulfills the* *following continuity properties:*

(i) *Times T_1 and T_2 are Lipschitz continuous functions of initial $x_{2,0}$.*

(ii) *The operator $\mathcal{O}$ is Lipschitz continuous and takes values in J.*

Proof. For a given $x_{2,0} \in (0, s_{in} - x_1^r]$, denote by $x(\cdot)$ the unique solution of (5.3) for the initial condition (x_1^r, x_{02}) and the time-varying $D(\cdot)$ given by Proposition 3.1.

Let us prove (i). For $t \in [0, T_1]$, one has $D(t) = D_M$ and T_1 is defined as the first time that $x(\cdot)$ reaches the set

$$\mathcal{T}_1 := \{x \in \mathcal{S}\ ;\ x_1 \leq \bar{x}_1 \text{ and } x_2 \leq \varepsilon\}.$$

For $t \in (T_1, T_2)$, one has $D(t) = D_m$. We know that there exists $\bar{T}_1 \in (T_1, T_1 + \check{T})$ such that $x(\cdot)$ crosses the line $\{x_1 = x_1^r\}$ at time $\bar{T}_1$ with $x_2(\bar{T}_1) < \check{x}_{2,0}$ (see step 2 of the proof of Proposition 3.1). Therefore, the state $x(\bar{T}_1)$ is below the point $\check{P}^r$. As $x(\cdot)$ cannot cross the orbit of $\check{P}^r$ (denoted $\check{\gamma}$) on $(\bar{T}_1, T_2)$, $x(\cdot)$ crosses the line $\{x_1 = x_1^r\}$ at some time T_2 such that

$$x_2(T_2) > \underline{x}_2. \tag{5.16}$$

Hence, T_2 can be defined as the first time $t > T_1 + \check{T}$ such that $x(\cdot)$ reaches the set

$$\mathcal{T}_2 := \{x \in \mathcal{S}\ ;\ x_1 \leq x_1^r \text{ and } x_2 \geq \underline{x}_2\},$$

at time t. On the positive set $\mathcal{S}^+ := \{x \in \mathcal{S}\ ;\ x_1 > 0\ ;\ x_2 > 0\}$, that is invariant by (5.3), we are in a position to introduce the first entry time functions:

$$R_1(x_0) := \inf\{t \geq 0\ ;\ z(t, x_0, D_M) \in \mathcal{T}_1\},\ R_2(x_0) := \inf\{t \geq 0\ ;\ z(t, x_0, D_m) \in \mathcal{T}_2\},$$

with $x_0 \in \mathcal{S}^+$. Then, for $x_{2,0} \in (0, s_{in} - x_1^r]$, Proposition 3.1 allows to write the composition

$$\mathcal{O}(x_{2,0}) = z_2(T_2 - T_1 - \check{T}, x(T_1 + \check{T}), D_m), \tag{5.17}$$

with

$$\begin{cases} T_1 & := & R_1(x_1^r, x_{2,0}), \\ x(T_1) & := & z(T_1, (x_1^r, x_{2,0}), D_M), \\ x(T_1+\check{T}) & := & z(\check{T}, x(T_1), D_m), \\ T_2 & := & T_1 + \check{T} + R_2(x(T_1+\check{T})), \end{cases}$$

thanks to the definitions of T_1, T_2, and $\check{T}$. From the continuous dependency of an ODE w.r.t. initial conditions, (see, *e.g.*, [Per13]), the maps $x_0 \mapsto z(t, x_0, D)$ (for a fixed t) and $t \mapsto z(t, x_0, D)$ (for a fixed x_0) are Lipschitz continuous, given a constant $D \in [D_m, D_M]$. Therefore, proving the Lipschitz continuity of T_1 and T_2 w.r.t $x_{2,0}$ essentially requires to prove the Lipschitz continuity of R_1 and R_2 over the set $\mathcal{S}^+$. Notice first that for constant $D = D_M$, resp. $D = D_m$, any solution in $\mathcal{S}^+$ converges asymptotically to a steady state that belongs to the interior of $\mathcal{T}_1$, resp. $\mathcal{T}_2$. Therefore, the targets $\mathcal{T}_1$ and $\mathcal{T}_2$ can be reached in a finite horizon from any initial condition in $\mathcal{S}^+$, and thus, R_1, R_2 are well defined with finite values in $\mathcal{S}^+$. To prove their Lipschitz continuity, we shall use Theorem 6.1 recalled in Appendix 2, showing that the inward pointing condition (5.22) is fulfilled on the boundary of $\mathcal{T}_1$ and $\mathcal{T}_2$ in $\mathcal{S}^+$.

Lipschitz continuity of R_1. Observe first that $\mathcal{T}_1$ is convex, hence the (convex) normal cone to $\mathcal{T}_1$ at some point $x \in \mathcal{S}^+$ of its boundary is given by the expression

$$N_{\mathcal{T}_1}(x) = \begin{cases} \mathbb{R}_+ \times \{0\}, & x_1 = \bar{x}_1, \ x_2 < \varepsilon, \\ \{0\} \times \mathbb{R}_+, & x_1 < \bar{x}_1, \ x_2 = \varepsilon, \\ \mathbb{R}_+ \times \mathbb{R}_+, & x_1 = \bar{x}_1, \ x_2 = \varepsilon. \end{cases}$$

Then, we can easily check that

$$\{x_1 = \bar{x}_1, x_2 \leq \varepsilon\} \ \Rightarrow \ f_1(x, D_M) < \phi_1 := (\mu_1(s_{in}) - D_M)\bar{x}_1 < 0,$$

$$\{x_1 \leq \bar{x}_1, x_2 = \varepsilon\} \ \Rightarrow \ f_2(x, D_M) < \phi_2 := (\mu_2(s_{in} - \varepsilon) - D_M)\varepsilon < 0.$$

From the preceding inequalities, we deduce that for any point x on the boundary of $\mathcal{T}_1$ in $\mathcal{S}^+$, one has

$$f(x, D_M) \cdot \nu < \min(\phi_1, \phi_2)\|\nu\|, \quad \nu \in N_{\mathcal{T}_1}(x) \setminus \{0\}.$$

This allows us to conclude that R_1 is Lipschitz continuous over $\mathcal{S}^+$, thanks to Theorem 6.1.

Lipschitz continuity of R_2. Observe that $\mathcal{T}_2$ is also convex, hence the (convex) normal cone to $\mathcal{T}_2$ at some point $x \in \mathcal{S}^+$ of its boundary is given by the expression

$$N_{\mathcal{T}_2}(x) = \begin{cases} \mathbb{R}_+ \times \{0\}, & x_1 = x_1^r, x_2 > \underline{x}_2, \\ \{0\} \times \mathbb{R}_-, & x_1 < x_1^r, x_2 = \underline{x}_2, \\ \mathbb{R}_+ \times \mathbb{R}_-, & x_1 = x_1^r, x_2 = \underline{x}_2. \end{cases}$$

One can easily that the following properties are fulfilled:

$$\begin{aligned} \{x_1 = x_1^r, x_2 > \underline{x}_2\} &\Rightarrow f_1(x, D_m) < \psi_1 := f_1((x_1^r, \underline{x}_2), D_m), \\ \{x_1 < x_1^r, x_2 = \underline{x}_2\} &\Rightarrow f_2(x, D_m) > \psi_2 := f_2((x_1^r, \underline{x}_2), D_m), \\ \{x_1 = x_1^r, x_2 = \underline{x}_2\} &\Rightarrow f(x, D_m) \cdot \nu = \psi_1 \nu_1 + \psi_2 \nu_2. \end{aligned}$$

As a consequence of (5.14), (5.15) and (5.16), note that one has $x_1^m - x_1^r < \underline{x}_2 < x_2^m - x_1^r$. This allows us to conclude that $\psi_1 < 0$ and that $\psi_2 > 0$ yielding the inequality

$$f(x, D_m) \cdot \nu \le \min(\psi_1, -\psi_2) \|\nu\|, \quad \nu \in N_{\mathcal{T}_2}(x) \setminus \{0\},$$

for any x on the boundary of the set $\mathcal{T}_2$ in $\mathcal{S}^+$. This proves the Lipschitz continuity of R_2 on $\mathcal{S}^+$ using again Theorem 6.1. We conclude that both T_1 and T_2 are Lipschitz continuous w.r.t $x_{2,0}$.

Let us prove now (ii). Recall that $\mathcal{O}$ can be written as function of T_1 and T_2 (see (5.17)). As T_1 and T_2 are Lipschitz continuous w.r.t $x_{2,0}$ then $\mathcal{O}$ as well. Combining (5.14) and (5.16) gives that $\mathcal{O}(x_{2,0}) \in J$ as was to be proved. □

Remark 4.1. *Since T_1 and T_2 are continuous then times T_i, $i \ge 3$, introduced in Proposition 3.1 are also continuous functions of $x_{2,0}$.*

Let us now study the monotonicity of the operator $\mathcal{O}$.

Proposition 4.2. *The operator $\mathcal{O}$ is decreasing.*

Proof. Take two points $x_{2,0}^{\pm} \in (0, s_{in} - x_1^r]$ such that $x_{2,0}^- < x_{2,0}^+$, and let us show that $\mathcal{O}(x_{2,0}^-) > \mathcal{O}(x_{2,0}^+)$. Denote by $x^+(\cdot)$, $x^-(\cdot)$ the solutions generated by the time-varying $D(\cdot)$ given by Proposition 3.1 and for the initial conditions $x^+(0) = (x_1^r, x_{2,0}^+)$ and $x^-(0) = (x_1^r, x_{2,0}^-)$ respectively. One can then write $x^+(0) \preccurlyeq x^-(0)$. For convenience, we denote by T_1^+, T_2^+ and T_1^-, T_2^- the times T_1, T_2 (as in Proposition 3.1) associated with $x^+(\cdot)$ and $x^-(\cdot)$ respectively, and let us set

$$\bar{T}_1^+ := \inf\{t > T_1^+ \ ; \ x_1^+(t) > x_1^r\} \quad \text{and} \quad \bar{T}_1^- := \inf\{t > T_1^- \ ; \ x_1^-(t) > x_1^r\}.$$

First, let us note that the time-varying removal rate given by Proposition 3.1 satisfies $D(t) = D_M$ for both trajectories in a neighborhood of $t = 0$. Using the order property (5.9), one can then write

$$z(t, x^+(0), D_M) \preccurlyeq z(t, x^-(0), D_M), \quad t \geq 0. \tag{5.18}$$

Thanks to this property, we must have $T_1^+ \geq T_1^-$ (otherwise, x^+ reaches $\mathcal{E}$ at some time $T_1^+ < T_1^-$ implying a contradiction with (5.18)). Therefore one gets (recall that trajectories with $D = D_M$ decrease), we obtain the inequality

$$\begin{array}{rcl} x_1^+(T_1^+) & = & z_1(T_1^+, x^+(0), D_M) \\ & \leq & z_1(T_1^-, x^+(0), D_M) \\ & \leq & z_1(T_1^-, x^-(0), D_M) \\ & = & x_1^-(T_1^-). \end{array}$$

Since the orbits of (5.3) with $D = D_M$ do not intersect, we also obtain that $x_2^+(T_1^+) \geq x_2^-(T_1^-)$ which implies that $x^+(T_1^+) \preccurlyeq x^-(T_1^-)$. Because $x_1^+(T_1^+) < x_1^-(T_1^-)$, the time needed by $x^+(\cdot)$ to reach the liner $\{x_1 = x_1^r\}$ from $x_1^+(T_1^+)$ is greater than the time of $x^-(\cdot)$ to reach the line $\{x_1 = x_1^r\}$ from $x_1^-(T_1^-)$. This gives $\bar{T}_1^+ - T_1^+ \geq \bar{T}_1^- - T_1^-$.

We now consider $x^+(T_1^+)$ and $x^-(T_1^-)$ as initial conditions for (5.3) with $D = D_m$. Then one gets

$$z(t, x^+(T_1^+), D_m) \preccurlyeq z(t, x^-(T_1^-), D_m), \quad t \geq 0.$$

In the same way as previously, we deduce the inequality:

$$\begin{array}{rcl} x_2^+(\bar{T}_1^+) & = & z_2(\bar{T}_1^+ - T_1^+, x^+(T_1^+), D_m) \\ & \geq & z_2(\bar{T}_1^- - T_1^-, x^+(T_1^+), D_m) \\ & \geq & z_2(\bar{T}_1^- - T_1^-, x^-(T_1^-), D_m) \\ & = & x_2^-(\bar{T}_1^-). \end{array}$$

Since the orbits of (5.3) with $D = D_m$ do not intersect, one necessarily has $x_2^+(\bar{T}_1^+) > x_2^-(\bar{T}_1^-)$ (and also $x_1^+(\bar{T}_1^+) = x_1^-(\bar{T}_1^-)$). Finally, D is constant equal to D_m for both trajectories until the first instant at which one of the two trajectory leaves the set $\mathcal{K}(x_1^r)$. Since $x^+(\bar{T}_1^+)$ is above $x^-(\bar{T}_1^-)$, the point $x^-(T_2^-)$ is necessarily above $x^+(T_2^+)$, that is

$$x_2^-(T_2^-) > x_2^+(T_2^+),$$

which implies the desired inequality $\mathcal{O}(x_{2,0}^+) < \mathcal{O}(x_{2,0}^-)$. □

4.2 Existence and attractivity of periodic solutions

In this section, we study how for any initial condition, the time-varying removal rate $D(\cdot)$ given in Proposition 3.1 allow system (5.3) to synchronize with a periodic solution (*i.e.* any solution of (5.3) associated with $D(\cdot)$ converges asymptotically to a periodic solution).

4.2.1 Existence of periodic solutions

The existence of a weakly resilient periodic trajectory follows from the previous results about the operator $\mathcal{O}$.

Corollary 4.1. *There exists a unique positive periodic solution $x^\star(\cdot)$ associated with the time-varying $D(\cdot)$ given by Proposition 3.1 such that $x^\star(0) = (x_1^r, x_{2,0}^\star)$ with $x_{2,0}^\star$ satisfying $\mathcal{O}(x_{2,0}^\star) = x_{2,0}^\star$.*

Proof. Consider the function $\varphi : (0, s_{in} - x_1^r] \to \mathbb{R}$ defined as

$$\varphi(x_2) := \mathcal{O}(x_2) - x_2, \quad x_2 \in (0, s_{in} - x_1^r].$$

From Propositions 4.1 and 4.2, $\varphi(\cdot)$ is continuous and decreasing. Moreover, it verifies $\varphi(x_2) > 0$ for $x_2 < \underline{x}_2$ and $\varphi(x_2) < 0$ for $x_2 > \bar{x}_2$ (because $\mathcal{O}$ is with values in J). We can then conclude that $\varphi(\cdot)$ possesses a unique zero in the interval $(0, s_{in} - x_{1,r}]$, or equivalently that there exists a unique fixed point $x^\star_{2,0}$ of $\mathcal{O}$. The solution $x^\star(\cdot)$ for the initial condition $(x_1^r, x^\star_{2,0})$ verifies $x^\star(T_2^\star) = x^\star(0)$, where $T_2^\star$ is equal to the time T_2 generated by the time-varying $D(\cdot)$ given in Proposition 3.1, and is thus $T_2^\star$-periodic. We conclude that $x^\star(\cdot)$ is the unique periodic solution such that $x_1^\star(0) = x_1^r$ and $\mathcal{O}(x_2^\star(0)) = x_2^\star(0)$. □

4.2.2 Attractivity of the periodic solution

Due to the particular structure of the non-autonomous dynamics (the times T_i are not known explicitly), it appears that determining explicitly a bound on the Lipschitz rank of $\mathcal{O}$ is quite difficult. However, in all the simulations we performed, the operator $\mathcal{O}$ appears to be contractive, providing ε to be sufficiently small. We thus posit the following conjecture.

Conjecture 4.1. *For ε sufficiently small, the operator $\mathcal{O}$ is contractive on J.*

Proposition 4.3. *Under the conjecture, for ε sufficiently small an any initial condition in $\mathcal{K}(x_1^r)$, the solution $x(\cdot)$ generated by the time-varying $D(\cdot)$ given in Proposition 3.1 converges asymptotically to the periodic solution $x^\star(\cdot)$ up to a time shift $\bar{\sigma}$:*

$$\lim_{t \to +\infty} x(t + \bar{\sigma}) - x^\star(t) = 0.$$

Proof. Fix an initial condition in $\mathcal{K}(x_1^r)$. From Proposition 3.1 we know that the solution $x_1(\cdot)$ reaches x_1^r in finite time. Let t_0 be the first time t for which $x_1(t_0) = x_1^r$ and let $x_{2,0} = x_2(t_0)$. We can then consider, without any loss of generality, $(x_1^r, x_{2,0})$ as initial condition. Let T_i be the sequence of times given by Proposition 3.1. The trajectory $x(\cdot)$ is then solution of the non-autonomous dynamics $\dot{x} = F(t, x)$ with

$$F(t,x) = \begin{cases} f(x, D_M), \ t \in [T_{2k}, T_{2k+1}) \\ f(x, D_m), \ t \in [T_{2k+1}, T_{2k+2}) \end{cases} \qquad (k \in \mathbb{N})$$

As $\mathcal{O}$ is contractive, one has the following limit

$$\lim_{k\to+\infty} x_2(T_{2k}) = \lim_{k\to+\infty} \mathcal{O}^k(x_{2,0}) = x^\star_{2,0}. \tag{5.19}$$

Let $T_i^\star$ be the sequence of times given by Proposition 3.1 for the initial condition $x_0^\star = (x_1^r, x^\star_{2,0})$ and posit

$$F^\star(t,x) = \begin{cases} f(x, D_M), \ t \in [T^\star_{2k}, T^\star_{2k+1}) \\ f(x, D_m), \ t \in [T^\star_{2k+1}, T^\star_{2k+2}) \end{cases} \qquad (k \in \mathbb{N})$$

Clearly, on has $T^\star_{i+2} = T^\star_i + T^\star_2$ i.e. $F^\star$ is a $T^\star_2$-periodic dynamics.

We define now the following functions

$$\tilde{F}(\tau, x) = \frac{T_{i+1} - T_i}{T^\star_{i+1} - T^\star_i} F(\tau, x), \quad \tau \in [T^\star_i, T^\star_{i+1})$$

$$g(t) = T^\star_i + \frac{T^\star_{i+1} - T^\star_i}{T_{i+1} - T_i}(t - T_i), \quad t \in [T_i, T_{i+1})$$

Clearly, the solution $x(\cdot)$ of $\dot{x} = F(t,x)$ satisfies $x(t) = \tilde{x}(g(t))$ for any $t \geq 0$, where $\tilde{x}(\cdot)$ is solution of

$$\frac{d\tilde{x}}{d\tau}(\tau) = \tilde{F}(\tau, \tilde{x}(\tau)), \quad \tilde{x}(0) = x(0).$$

Thanks to the continuity property of T_i (see Proposition 4.1 and Remark 4.1) and (5.19), one concludes that

$$\lim_{k\to+\infty} (T_{2k+i} - T_{2k}) = \lim_{k\to+\infty} T_i(x_2(T_{2k})) = T_i(x^\star_{2,0}), \ i = 1, 2, \tag{5.20}$$

which gives

$$\lim_{k\to+\infty} (T_{2(k+1)} - T_{2k}) = T_2(x^\star_{2,0}) = T^\star_2, \tag{5.21}$$

and

$$\lim_{k\to+\infty} (T_{2k+i+1} - T_{2k+i}) = T_{i+1}(x^\star_{2,0}) - T_i(x^\star_{2,0}), \ i = 0, 1.$$

Then, one has

$$\lim_{i\to+\infty} \frac{T_{i+1} - T_i}{T^\star_{i+1} - T^\star_i} = 1$$

and deduce that $\tilde{F}$ is an asymptotically periodic dynamics with $F^\star$ as limit (see Definition 7.1 in Appendix 3). Moreover, one has

$$\lim_{k\to+\infty} \tilde{x}(kT_2^\star) = \lim_{k\to+\infty} \tilde{x}(T_{2k}^\star) = \lim_{k\to+\infty} x(T_{2k}) = x_0^\star.$$

Therefore, we can apply the Theorem 7.1 (from [Zha96], recalled in Appendix 3) which gives

$$\lim_{t\to+\infty} (\tilde{x}(t) - x^\star(t)) = 0$$

and we have obtained thus the following limit

$$\lim_{t\to+\infty} (x(t) - x^\star(g(t))) = 0.$$

We show now that the time shift $\sigma(t) := t - g(t)$ admits a limit $\bar{\sigma}$. Notice that one has $\sigma(T_i) = T_i - T_i^\star$ for any i. Therefore, it is enough to prove that the sequence $u_k := T_{2k} - T_{2k}^\star$ converges.

As $\mathcal{O}$ is contractive on J, there exists $\alpha \in (0,1)$ such that

$$|x_2(T_{2(k-1)}) - x_{2,0}^\star| = |\mathcal{O}(x_2(T_{2(k-2)}) - x_{2,0}^\star| \leq \alpha^{k-1}|x_{2,0} - x_{2,0}^\star|, \quad k > 1$$

As the map $x_{2,0} \mapsto T_2$ is Lipschitz continuous (cf Proposition 4.1), say of rank L, one obtains

$$|u_k - u_{k-1}| = |T_2(x_2(T_{2(k-1)}) - T_2^\star| \leq L\alpha^{k-1}|x_{2,0} - x_{2,0}^\star|$$

and let us finally show that u_k is a Cauchy sequence. For any $n > 1$ and $k > 1$, one has

$$|u_{n+k} - u_n| \leq L \sum_{i=0}^{k-1} \alpha^{n+i}|x_{2,0} - x_{2,0}^\star| = L\alpha^n \frac{1-\alpha^k|}{1-\alpha}|x_{2,0} - x_{2,0}^\star|$$

As $\alpha < 1$, on obtains that $|u_{n+k} - u_n|$ tends to 0 when n tends to ∞ uniformly in k. We conclude that the Cauchy sequence u_k admits a limit, which gives the asymptotic shift $\bar{\sigma}$. □

5 Appendix 1: list of notations

We remind in the next table all the parameters used in Section 3 and 4.

	Notation	Definition
$D = D_M$	$\hat{P}^r$	$(x_1^r, s_{in} - x_1^r)$
	$\hat{x}(\cdot)$	$z(\cdot, \hat{P}^r, D_M)$
	$\hat{\gamma}^+$	$\{\hat{x}(t)\ ;\ t \geq 0\}$
	$\hat{t}$	$\inf\{t > 0\ ;\ \hat{x}_1(t) < \bar{x}_1\}$
	δ	$\hat{x}_2(\hat{t})$
$D = D_m$	$\check{x}_{2,0}$	$\check{x}_2(0)$ with $\check{x}_2(0) < \min(x_1^m - x_1^r, \delta)$
	$\check{P}^r$	$(x_1^r, \check{x}_{2,0})$
	$\check{x}(\cdot)$	$z(\cdot, \check{P}^r, D_m)$
	$\check{\gamma}^-$	$\{\check{x}(t)\,;\, t \leq 0\}$
	$\check{\gamma}$	$\{\check{x}(t)\,;\, \forall t\}$
	$\bar{x}_2$	$W^u(E_1^m) \cap \{x_1 = x_1^r\}$
	η	$x_1^m - x_1^r - \check{x}_{2,0}$
	P_ε	$(\underline{x}_1, \varepsilon)$ intersection between $\hat{\gamma}^+$ and $\check{\gamma}^-$
	$\hat{T}$	such that $\hat{x}(\hat{T}) = P_\varepsilon$
	$\check{T}$	such that $\check{x}(-\check{T}) = P_\varepsilon$
	$\mathcal{E}$	$(0, \bar{x}_1] \times (0, \varepsilon]$
	$\check{Q}^r$	$(x_1^r, \underline{x}_2)$ intersection, different from $\check{P}^r$, between $\check{\gamma}$ and $\{x_1 = x_1^r\} \cap \mathcal{S}$

6 Appendix 2: Petrov's condition

We recall here a result about the continuity of the first entry time function (see Theorem 8.25 in [CS04]), that is stated here for a non-controlled dynamics. Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a mapping of class C^1 with linear growth, and denote by $y(\cdot, y_0)$ the unique solution of the Cauchy problem:

$$\begin{cases} \dot{y} &= g(y), \\ y(0) &= y_0, \end{cases}$$

defined over $\mathbb{R}_+$. Hereafter, we are given a non-empty compact subset $\mathcal{T}$ of $\mathbb{R}^n$ and for $y \in \mathcal{T}$, the set $N^p_{\mathcal{T}}(y)$ stands for the *proximal normal cone* to the set $\mathcal{T}$ at the point y (see [Cla13]). The standard inner product is written $a \cdot b$ for $a, b \in \mathbb{R}^n$, and $\|a\|$ denotes the euclidean norm of the vector a.

Theorem 6.1. *Suppose that the Petrov condition*

$$\exists \gamma < 0,\ \forall y \in \partial\mathcal{T},\ \forall \nu \in N^p_{\mathcal{T}}(y) \setminus \{0\}, \quad g(y) \cdot \nu < \gamma \|\nu\|, \tag{5.22}$$

is fulfilled. Then, the first entry time function

$$R(y_0) := \inf \{t \geq 0 \ ; \ y(t, y_0) \in \mathcal{T}\}, \quad y_0 \in \mathbb{R}^n,$$

is Lipschitz continuous in its open domain $\{y_0 \in \mathbb{R}^n \ ; \ R(y_0) < +\infty\}$*.*

If $\mathcal{T}$ is convex, the set $N^p_{\mathcal{T}}(y)$ coincides with the *convex normal cone* (see, *e.g.*, [Cla13]) defined for $y \in \mathcal{T}$ as:

$$N_{\mathcal{T}}(y) := \{q \in \mathbb{R}^n \ ; \ q \cdot (z - y) \leq 0, \quad \forall z \in \mathcal{T}\}.$$

7 Appendix 3: Asymptotically periodic systems

We recall first the definition of asymptotically periodic semi-flows in $\mathbb{R}^n$.

Definition 7.1. *A non-autonomous semiflow* Φ*:* $\{(t, s); \ 0 \leq s \leq t < \infty\} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ *is asymptotically periodic with limit* ω*-periodic semi-flow* $T(t)$*:* $\mathbb{R}^n \mapsto \mathbb{R}^n$*,* $t \geq 0$ *if*

$$\Phi(t_j + n_j\omega, n_j\omega, x_j) \to T(t)x, \ j \to \infty$$

for any sequences $t_j \to t$*,* $n_j \to \infty$*,* $x_j \to x$ *when* $j \to \infty$*, with* x*,* x_j *in* $\mathbb{R}^n$*.*

The following result can be found in [Zha96] (Theorem 3.1).

Theorem 7.1. *Let* Φ*:* $\{(t, s); \ 0 \leq s \leq t < \infty\} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ *be an asymptotically periodic semi-flow with limit* ω*-periodic semi-flow* $T(t)$*:* $\mathbb{R}^n \mapsto \mathbb{R}^n$*,* $t \geq 0$*. Denote* $T_n(x) = \phi(n\omega, 0, x)$ *and* $S(x) = T(\omega)x$*,* $n \geq 0$*,* $x \in \mathbb{R}^n$*. If* A_0 *is a compact subset* *of* $\mathbb{R}^n$ *invariant by the semi-flow* S *and* $y \in \mathbb{R}^n$ *is such that* $d(T_n(t), A_0) \to 0$ *when* $n \to \infty$ *then*

$$\lim_{t \to +\infty} d(\Phi(t, 0, y), T(t)A_0) = 0.$$

CHAPITRE 6

A HYBRID CONTROL AGAINST SPECIES INVASION IN THE CHEMOSTAT

Ce travail est à paraître en ligne dans les *proceedings of the* 58^{th} *IEEE Conference on Decision and Control*.

Contents

1 introduction

The well known chemostat model [SW95, HLRS17] describes the microbial growth in continuous culture [Pan95] and is also often used to represent continuously stirred tank reactors [BD90]:

$$\begin{array}{rcl} \dot{s} & = & -\frac{1}{Y}\mu(s)x + D(s_{in} - s), \\ \dot{x} & = & (\mu(s) - D)x. \end{array} \tag{6.1}$$

The variables s and x denote the concentrations of substrate and biomass, respectively. The operating parameters are the input concentration of substrate s_{in} and the removal rate D (we assume that the effective removal rate is the same on both concentrations, which amounts to consider that mortality or attachment of bacteria can be neglected). The parameter Y and the function $\mu(\cdot)$ are the yield conversion and the specific growth rate, that we assume here to be increasing with $\mu(0) = 0$ (as it is often the case in microbiology [Pan95]). Without loss of generality, one can take $Y = 1$ (at the price to change the biomass unit). In many industrial applications, bacteria are expected to provide ecosystem services (such as water purification) or produce substances of economic interest (in pharmacology for instance). It is often of primer importance to maintain the concentration of the biomass x at or above a given threshold. One can easily check that a positive equilibrium $(s^\star, x^\star)$ of (6.1) verifies $\mu(s^\star) = D$ and $x^\star = Y(s_{in} - s^\star)$. Typically, bioreactors are operated at steady state choosing the removal rate D to be equal to a nominal value $D^\star := \mu(s_{in} - x^\star/Y)$ where $x^\star$ is the desired biomass concentration (see for instance [BD90, HLRS17]). However, it is also well known that open systems are subject to bacterial contamination or genetic evolution. Therefore, another bacterial species can appear in the chemostat system at any time. The chemostat model with more than one species (here two, assuming without loss of generality that the yield coefficients Y_i are equal to 1):

$$\left\{ \begin{array}{l} \dot{s} = -\mu_1(s)x_1 - \mu_2(s)x_2 + D(s_{in} - s), \\ \dot{x_1} = (\mu_1(s) - D)x_1, \\ \dot{x_2} = (\mu_2(s) - D)x_2, \end{array} \right. \tag{6.2}$$

has been widely studied in the literature [AH77, SW95, Gro97, JP01, HLRS17]. The main result is the so-called "Competitive Exclusion Principle" [HHW77, AM80, BW85], which states that (generically) at most one species survives at steady state. Then, depending on the characteristics of the invading species, this one can settle and eradicate the resident species, which can be catastrophic for operators if the new species does not provide similar ecosystem services or by-products. However, competing species often present compromises. Typically, a species could be specialized for low substrate concentrations while another one could be for larger concentrations. In such cases, it has been shown that applying a periodic time-varying removal rate $D(\cdot)$ could allow coexistence of both species [Smi81, BHW85, WZ+98]. It appears that the design of such time-varying controls requires the perfect knowledge of the growth functions $\mu_i(\cdot)$. In addition, to our best knowledge, the performances in terms of services that continue to be provided by the resident species, have not been studied in the literature. This is precisely the objective of the present work. In particular, we shall see that hybrid controllers can be defined without a precise knowledge of the functions $\mu_i(\cdot)$ allowing coexistence of both species. In addition, for the corresponding solution, the time spent by the density of the resident species above a given threshold is infinite.

The paper is organized as follows. In Section 2, we setup assumptions and study the resilience of the chemostat model (6.1) in presence of an invading species, considering system (6.2). We introduce here a new concept of *weak resilience*. In Section 3, we present recent results about the design of time-varying removal rates that make the system weak resilient. We then show that such open-loop controls can indeed be synthesized by a hybrid feedback controller. Section 4 illustrates with numerical simulations the behavior of the proposed controller and discusses the role of its parameters.

2 Resilience analysis

For system (6.2), let us consider the following (classical) assumptions:

Hypothesis 2.1. *The functions $\mu_i(\cdot)$ ($i = 1, 2$) are C^1 increasing functions with*

$\mu_i(0) = 0$.

Let us also introduce the *break-even concentrations*:

$$\lambda_i(D) := \sup\left\{s \in [0, s_{in}]\ ;\ \mu_i(s) < D\right\}, \quad i = 1, 2.$$

Then, the Competitive Exclusion Principle (CEP) [BW85, SW95, HLRS17] states as follows.

Theorem 2.1 (CEP). *Under Assumption 2.1, if* $\lambda_i(D) < \min(\lambda_j(D), s_{in})$ *for* $i \neq j$, $i, j \in \{1, 2\}$, *then any positive solution of* (6.2) *converges asymptotically to the equilibrium* $(s^\star, x_1^\star, x_2^\star)$ *with* $s^\star = \lambda_i(D)$, $x_i^\star = s_{in} - \lambda_2(D)$ *and* $x_j^\star = 0$.

Therefore, several cases appear:

- if $\mu_2(s) > \mu_1(s)$ for any $s > 0$, then one has $\lambda_2(D) < \lambda_1(D)$ for any $D \in (0, \mu_1(s_{in}))$, and the resident species never survives under invasion by species 2.
- On the opposite, if $\mu_2(s) < \mu_1(s)$ for any $s > 0$, the resident species is safe in presence of such an invader (with $D < \mu_1(s_{in})$ to avoid the washout equilibrium).
- However, as mentioned in the introduction, it often happens that species are complementary: if the invader can settle and eradicate the resident species for small values of s, the issue of the competition is reversed under large concentrations of s, what we shall consider in the sequel with the following assumption:

Hypothesis 2.2. *There exists* $\bar{s} \in (0, s_{in})$ *such that*

$$(\mu_1(s) - \mu_2(s))(s - \bar{s}) > 0, \quad s \in [0, s_{in}] \text{ and } s \neq \bar{s}. \tag{6.3}$$

Denote then $\bar{D} := \mu_1(\bar{s}) = \mu_2(\bar{s})$.

For any $D \in (\bar{D}, \mu_1(s_{in}))$, the resident species 1 wins the competition and its density at steady state is equal to $x_1^* := s_{in} - \lambda_1(D)$. Therefore the number

$$\bar{x}_1 := s_{in} - \lambda_1(\bar{D}) = s_{in} - \bar{s} > 0.$$

is the largest density level that the system can maintain in presence of the invader 2 under constant removal rate. However, one may wonder if it is possible to maintain higher levels with time varying removal rate $D(\cdot)$ taking values in an interval $[D_m, D_M]$ with

$$0 < D_m < \bar{D} < \mu_1(s_{in}) < D_M. \tag{6.4}$$

One can easily check that the domain

$$\{(s, x_1, x_2) \in \mathbb{R}^3_+ \ ; \ s + x_1 + x_2 = s_{in}\},$$

is invariant by (6.2) and attractive for any persistently exciting control $D(\cdot)$. By persistently exciting control, we mean any non-negative function $D(\cdot)$ such that $\int_0^{+\infty} D(t)dt = +\infty$ (see [BD90]). Considering that (6.1) is already at steady state before invasion, and that the quantity x_2 at invasion is very small, we shall consider in the sequel the reduced dynamics on this domain, that is, the planar dynamics

$$\dot{x} = f(x, D) := \begin{bmatrix} (\mu_1(s_{in} - x_1 - x_2) - D)x_1 \\ (\mu_2(s_{in} - x_1 - x_2) - D)x_2 \end{bmatrix}, \tag{6.5}$$

defined on the invariant set

$$\mathcal{S} := \{x \in \mathbb{R}^2_+ \ ; \ x_1 + x_2 \leq s_{in}\}.$$

Consider then a threshold $x_1^r \in (\bar{x}_1, s_{in})$ satisfying

$$0 < D_m < \mu_1(s_{in} - x_1^r), \tag{6.6}$$

and define a subset $\mathcal{K}(x_1^r)$ of $\mathcal{S}$ as:

$$\mathcal{K}(x_1^r) := \{x \in \mathcal{S} \ ; \ x_1 \geq x_1^r \ \text{ and } \ x_2 > 0\}.$$

Let us now formulate the invasion problem in terms of *resilience*. Assume that, at the initial time, species 1 is alone with a density level equal to x_1^r ensured by the choice of the constant dilution rate $D^r := \mu_1(s_{in} - x_1^r) < \bar{D}$, and that species 2 (satisfying Assumption 2.2) appears. We then say that system (6.5) with the constant removal rate D^r is not resilient with respect to the set $\mathcal{K}(x_1^r)$ because the trajectory escapes from $\mathcal{K}(x_1^r)$ and never comes back (indeed, it converges to the equilibrium point $(\lambda_2(D^r), 0, s_{in} - \lambda_2(D^r))$.

Let us now study resilience when one allows time-varying removal rate, in terms of viability analysis. Let us recall that the viability kernel of a closed subset K of $\mathbb{R}^n$ for a controlled dynamics $\dot{x} = f(x, u(t))$ (with $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $u(t) \in U \subset \mathbb{R}^m$), is the largest closed subset V of K such that for any $x_0 \in V$ there exists $u(\cdot)$ for which the solution with $x(0) = x_0$ verifies $x(t) \in K$ for any $t \geq 0$ (see [Aub09]).

Lemma 2.2. *The viability kernel $Viab(cl\, \mathcal{K}(x_1^r))$ of the closure of the set $\mathcal{K}(x_1^r)$ for* (6.5) *with controls $D(\cdot) \in [D_m, D_M]$ (where D_m, D_M fulfill* (6.4) *and* (6.6)*) is given by*

$$Viab(cl\, \mathcal{K}(x_1^r)) = [x_1^r, s_{in}] \times \{0\}.$$

Proof. From (6.6), x_1^r is such that $x_1^r \in (\bar{x}_1, s_{in} - \lambda_1(D_m))$. Moreover, Assumptions 2.1, 2.2 and condition (6.4) also imply the inequality $0 < \lambda_2(D_m) < \lambda_1(D_m) < \bar{s}$. Take an initial condition $(x_{1,0}, x_{2,0})$ in $cl\, \mathcal{K}(x_1^r)$.

If $x_{2,0} = 0$, the solution of (6.5) verifies $x_2(t) = 0$ for any $t > 0$ and any control $D(\cdot)$. From (6.6), we can choose a constant control D such that $D_m \leq D < \mu_1(s_{in} - x_1^r)$, and we observe that (6.5) with $D(t) = D$ satisfies

$$x_1 = x_1^r \Rightarrow \dot{x}_1 = (\mu_1(s_{in} - x_1^r) - D)x_1^r > 0,$$

$$x_1 = s_{in} \Rightarrow \dot{x}_1 = -Ds_{in} < 0.$$

Thus, the constant control D allow the trajectory to stay in the desirable set $\mathcal{K}(x_1^r)$.

Assume now that $x_{2,0} > 0$. Then, a solution $x(\cdot)$ of (6.5) associated with an admissible control $D(\cdot)$ verifies $x_2(t) > 0$ for any time $t \geq 0$. Suppose by contradiction that $x(\cdot)$ stays in $\mathcal{K}(x_1^r)$ for any time $t \geq 0$. Then one has $s(t) < s_{in} - x_1^r < \bar{s}$ for any time $t \geq 0$. By Assumption 2.2, one has for any $t \geq 0$, $\mu_1(s(t)) - \mu_2(s(t)) < 0$ and thus, we deduce the inequality

$$\begin{aligned}\dot{s}(t) &> -\mu_2(s(t))x_1(t) - \mu_2(s(t))x_2(t) + D_m(s_{in} - s(t)) \\ &= (D_m - \mu_2(s(t)))(s_{in} - s(t)),\end{aligned}$$

for any time $t \geq 0$. Notice that any positive solution $\zeta(\cdot)$ of $\dot{\zeta} = (D_m - \mu_2(\zeta(t)))(s_{in} - \zeta(t))$ converges to $\lambda_2(D_m)$ when $t \to +\infty$. From (6.6) one has $\lambda_2(D_m) < s_{in} - x_1^r$, hence there exist $t_1 \geq 0$ and $\underline{s} \in (0, \lambda_2(D_m))$ such that one has $\zeta(t) > \underline{s}$ for any $t \geq t_1$. From the preceding inequality, we deduce that $s(\cdot)$ satisfies $s(t) \geq \zeta(t)$ for any time $t \geq 0$. We thus deduce that for any time $t \geq t_1$, one has $\mu_1(s(t)) - \mu_2(s(t)) \leq c$ with

$$c := \min\{\mu_1(\sigma) - \mu_2(\sigma) \; ; \; \sigma \in [\underline{s}, s_{in} - x_1^r]\} < 0.$$

If we differentiate the function $q_1 := x_1/x_2$ w.r.t. t, we find that

$$\dot{q}_1 = \Big(\mu_1(s(t)) - \mu_2(s(t))\Big) q_1,$$

with $s(t) = s_{in} - x_1(t) - x_2(t)$. One then obtains $\dot{q}_1 < c\, q_1$. Therefore q_1 decreases to zero and x_1 as well, leading to a contradiction. We conclude that the only solutions of (6.5) that stay in $cl\,\mathcal{K}(x_1^r)$ for any time are the ones starting with $x_{2,0} = 0$ as was to be proved. □

This lemma shows that no control strategy allows the resident species to stay in the desirable set $\mathcal{K}(x_1^r)$ in presence of the invader. The dynamics is thus not resilient for the domain $\mathcal{K}(x_1^r)$. We are now in a position to introduce a weaker concept of resilience.

Definition 2.1. *Let $x_1^r \in (\bar{x}_1, s_{in} - \lambda_1(D_m))$. The system* (6.5) *is said to be weakly resilient for the set $\mathcal{K}(x_1^r)$ if for any initial condition in $\mathcal{K}(x_1^r)$, there exists a control function $D(\cdot)$ with values in $[D_m, D_M]$ such that the corresponding solution of* (6.5) *satisfies*

$$\text{meas}\,\{t \geq 0 \; ; \; x(t) \in \mathcal{K}(x_1^r)\} = +\infty.$$

Such a function $D(\cdot)$ will be called a weakly resilient control.

The purpose of the next section is to construct explicitly such control functions.

3 The hybrid control

In this section, we shall consider bang-bang controls with values D_m or D_M (parameters D_m and D_M satisfying conditions (6.4) and (6.6)).

3.1 Construction of a weakly resilient control

Let us first give properties of the dynamical system (6.5) when D is constant ($D = D_m$ or $D = D_M$).

1. For $D = D_M$, the "washout" equilibrium $E_0 := (0, 0)$ is the single equilibrium in $\mathcal{S}$, which is globally asymptotically stable (see Fig. 6.1).

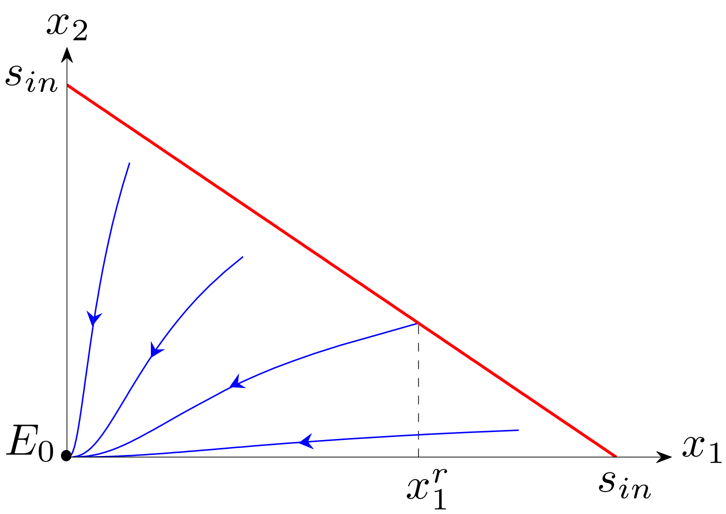

FIGURE 6.1 – *Phase portrait of* (6.5) *with the constant control* $D = D_M$.

2. For $D = D_m$: there are three equilibria in $\mathcal{S}$: E_0, $E_1 := (x_1^m, 0)$ and $E_2 := (0, x_2^m)$ with $x_i^m := s_{in} - \lambda_i(D_m)$. One can easily check that E_2 is a stable node while E_1 is a saddle point. By the Theorem of the stable and unstable manifolds, we deduce that the unstable manifold W^u of E_1 connects E_1 to E_2, and that the stable manifold of E_1 belongs to the x_2-axis (see Fig. 6.2).

One can also easily check that solutions of (6.5) with $D = D_M$ reach the

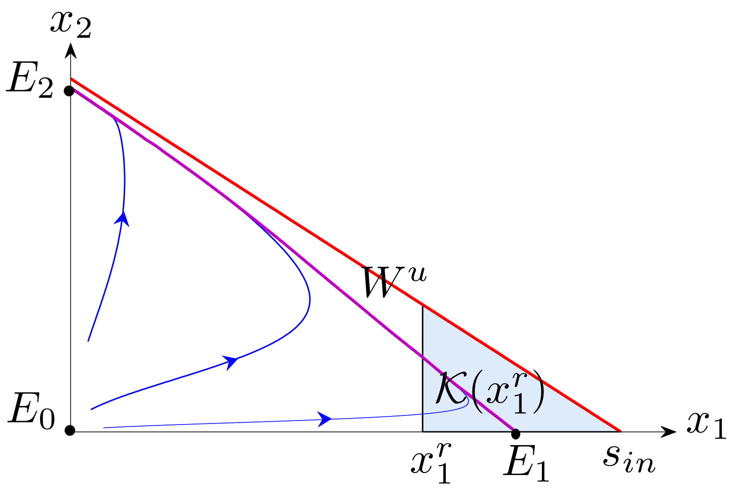

FIGURE 6.2 – *Phase portrait of* (6.5) *with the constant control* $D = D_m$.

origin tangentially to the x_2-axis. Therefore, if $D(t) = D_M$ for a sufficiently long time, the state will get so close to the x_2-axis that a switching to $D = D_m$ will produce a further trajectory that will remain close from the stable manifold of E_1 for a long time and that can then reach the set $\mathcal{K}(x_1^r)$, before moving towards the equilibrium E_2. Then, another switch to $D = D_M$ when the state leaves $\mathcal{K}(x_1^r)$ allows to come back again close to the origin, and so on.

The construction of a weakly resilient control precisely relies on asymptotic properties of (6.5) with a constant control $D = D_m$ or $D = D_M$. The next theorem is our main result and formalizes this methodology.

Theorem 3.1. *Assume that* D_m *and* D_M *satisfy* (6.4) *and* (6.6). *For any* $\varepsilon > 0$, *define the set*

$$\mathcal{E}_\varepsilon := (0, \bar{x}_1] \times (0, \varepsilon].$$

For ε *small enough and any initial condition* $x(0) \in \mathcal{K}(x_1^r)$, *there exists a piecewise constant function* $D(\cdot)$ *which alternates the values* D_M, D_m *on time intervals* $[T_i, T_{i+1})$, $i \in \mathbb{N}$ *satisfying:*

$$T_0 = 0 < T_1 < \cdots < T_i < T_{i+1} < \cdots \quad \text{and} \quad \lim_{i \to \infty} T_i = +\infty, \tag{6.7}$$

where $x(\cdot)$ *is the unique solution of* (6.5) *associated with* $D(\cdot)$ *such that*

(i) *if* $x(T_i) \notin \mathcal{E}_\varepsilon$, *one has* $D(t) = D_M$ *for* $t \in [T_i, T_{i+1})$ *with* T_{i+1} *defined as the first next entry time into* $\mathcal{E}_\varepsilon$.

(ii) *if* $x(T_i) \in \mathcal{E}_\varepsilon$*, one has* $D(t) = D_m$ *for* $t \in [T_i, T_{i+1})$ *and the trajectory* $x(\cdot)$ *enters into the set* $\mathcal{K}(x_1^r)$ *in finite time ;* T_{i+1} *is defined as the first next exit time from* $\mathcal{K}(x_1^r)$*.*

Moreover, $D(\cdot)$ *is a weakly resilient control.*

Du to lack of space, we do not provide the proof of this result (which is long and technical). It mainly consists in studying deeply properties (such as continuity, sign) of the operator:

$$\begin{array}{rccc} \mathcal{O}: & (0, s_{in} - x_1^r] & \to & (0, s_{in} - x_1^r] \\ & x_{2,0} & \mapsto & x_2(T_2) \end{array}$$

where $x(\cdot) = (x_1(\cdot), x_2(\cdot))$ denotes the unique solution of (6.5) for the initial condition $(x_1^r, x_{2,0})$ and the time-varying control $D(\cdot)$ given by Theorem 3.1, parameters D_m, D_M, ε being fixed. More details can be found in [BRT19].

Let us underline the fact that the control law given by Theorem 3.1 is an *open-loop* control, and that the computation of the switching times T_i requires the perfect knowledge of the functions $\mu_j(\cdot)$. In practice, an open-loop control is not robust with respect to the knowledge of the initial condition and the switching times. Moreover the characteristics of the invasive species is usually not known in advance. However, this control strategy cannot be written as a pure state feedback because when x_1 reaches the value x_1^r with $x_2 > \varepsilon$, one cannot decide if $D = D_m$ or $D = D_M$ without knowing the *past*, i.e., if x_1 is increasing or decreasing.

3.2 Synthesis with a hybrid control

Instead, we propose a hybrid controller associated with a logic-based switching algorithm represented by a piecewise signal $\sigma(\cdot)$ that can take three possible states denoted by χ_1, χ_2 and χ_3. The transitions between these three states is illustrated on Fig. 6.3.

One has the following result.

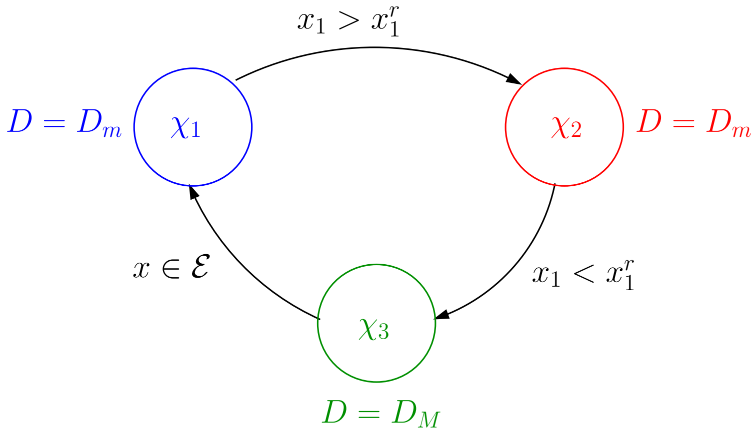

FIGURE 6.3 – *The logic-based algorithm associated with the hybrid control.*

Proposition 3.1. *Let $\sigma(\cdot) : \mathbb{R}_+ \to \{\chi_1, \chi_2, \chi_3\}$ be such that*

$$\begin{aligned} \sigma(t) &:= g(x_1(t), \sigma(t_-)) \\ &= \begin{cases} \chi_2 & \text{if } \sigma(t_-) = \chi_1 \text{ and } x_1(t) > x_1^r, \\ \chi_3 & \text{if } \sigma(t_-) = \chi_2 \text{ and } x_1(t) < x_1^r, \\ \chi_1 & \text{if } \sigma(t_-) = \chi_3 \text{ and } x(t) \in \mathcal{E}, \\ \sigma(t_-) & \text{otherwise}, \end{cases} \end{aligned}$$

with $\sigma(0) = \chi_3$. Then, the σ-feedback:

$$D(\sigma(t)) = \begin{cases} D_m & \text{if } \sigma(t) \in \{\chi_1, \chi_2\}, \\ D_M & \text{if } \sigma(t) = \chi_3. \end{cases} \tag{6.8}$$

guarantees the weak resilience of the dynamics with the control (6.8).

Proof. This automate is a simple way to memorize if the state x_1 is entering or leaving the set $\mathcal{K}(x_1^r)$. One can easily see that solutions of the coupled dynamics

$$\begin{aligned} \dot{x}(t) &= f(x(t), D(\sigma(t))), \quad x(0) \in \mathcal{K}(x_1^r), \\ \sigma(t) &= g(x_1(t), \sigma(t_-)), \quad \sigma(0) = \chi_3, \end{aligned}$$

(see for instance [Lib03] for the solution concept) are such that the times T_i given by Proposition 3.1 correspond exactly to the ones of σ (when switching from χ_2 to χ_3 or from χ_3 to χ_1). Provided that ε is small enough and that D_m and D_M satisfy (6.4) and (6.6), Proposition 3.1 then guarantees weak resilience of the hybrid controller (6.8). □

In [BRT19], it has been shown that there exist periodic solutions of (6.5) associated with the control given in Theorem 3.1. It has been conjectured that for ε sufficiently small, the operator $\mathcal{O} : x_{2,0} \mapsto x_2(T_2)$, where T_2 is given by Theorem 3.1 for the initial condition $(x_1^r, x_{2,0})$, is contractive. Then, under this condition, it has been proved that any trajectory with initial condition in $\mathcal{K}(x_1^r)$ and the weakly resilient control given by Theorem 3.1 converges asymptotically to a unique periodic solution $x^\dagger(\cdot)$ with a period $T^\dagger$, up to a time shift.

Remark 3.1. *The synthesis of the hybrid controller requires the single choice of the parameters D_m and D_M that satisfy* (6.4) *and* (6.6) *(which are quite loose conditions) and $\varepsilon > 0$ small enough. Therefore, it does not require the precise knowledge of the functions μ_i to ensure weak resilience against a species that fulfills Assumption A2, in contrast with more sophisticated approaches, such as model predictive control, which rely on more knowledge on the growth functions. Let us also underline that the controller switches between an environment that is unfavorable to both species ($D = D_M$) and an environment which is favorable to the invasive species ($D = D_m$). It is then not intuitive that the resident species could be dominant most of the time under such a switching. This property is strongly linked to the choice of ε that has simply to be sufficiently small, although we are not able to provide an explicit bound.*

4 Numerical simulations

Numerical simulations have been performed to illustrate the behavior of the trajectories generated by the hybrid controller and the role of its parameters on the proportion of time spent in the desired set $\mathcal{K}(x_1^r)$, as well as their impact on the productivity of the resident species.

We consider two growth functions of Monod's type:

$$\mu_1(s) := \frac{0.5s}{5+s} \; ; \; \mu_2(s) := \frac{0.16s}{0.13+s},$$

with $s_{in} = 5$, see Fig. 6.4.

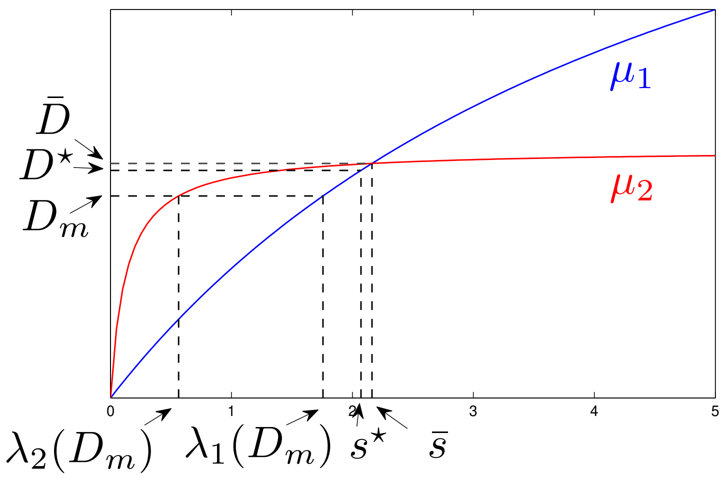

FIGURE 6.4 – *Graphs of the functions μ_1 and μ_2 with $s_{in} = 5$.*

Let us first notice that in absence of species 2, the hybrid controller gives $D(t) = D_m$. We have simulated an invasion of species 2 at time $t_I = 50$ with a sudden input of a small amount of concentration $x_2(t_I^+) = 0.2$. Before the invasion, one can observe that the system is at quasi-steady state and after the invasion, the control $D(t)$ alternates between D_m and D_M. The solution finally converges asymptotically to a periodic solution $x^\dagger(\cdot)$.

4.1 Proportion of time spent in $\mathcal{K}(x_1^r)$

On Figs. 6.5 and 6.6, two different values of ε have been chosen. In Table

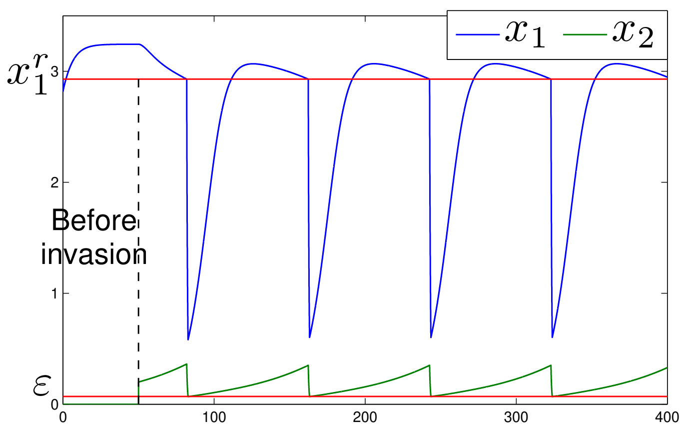

FIGURE 6.5 – *Time courses for $\varepsilon = 0.1$, $D_m = 0.13$ and $D_M = 2$.*

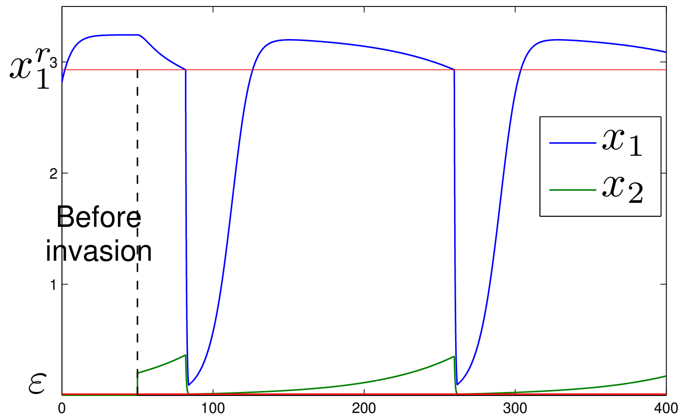

FIGURE 6.6 – *Time courses for $\varepsilon = 0.01$, $D_m = 0.13$ and $D_M = 2$.*

6.1, we also present for different values of ε, the proportion of time spent by the periodic solution $x^\dagger(\cdot)$ in the desirable set $\mathcal{K}(x_1^r)$, during one period (other parameters being fixed). When ε decreases, one observes that the period increases, as it requires a longer time for the trajectory to reach the set $\mathcal{E}$. Interestingly, one can also observe that the proportion of time spent in $\mathcal{K}(x_1^r)$ increases when ε decreases. Indeed, when ε is small, the trajectory gets close to the stable manifold of the saddle equilibrium E_1 when $D = D_m$ (which is the x_1-axis) and therefore the state stays a long time in the vicinity of E_1 which belongs to $\mathcal{K}(x_1^r)$.

ε	*period*	*% of time in $\mathcal{K}(x_1^r)$*	*average productivity*	*% of loss*
0.001	293	78.50	0.3562	15.51
0.01	177	75.20	0.3596	14.71
0.07	80.3	63.65	0.3674	12.86
0.15	40.5	36.67	0.3755	10.93

TABLE 6.1 – *Asymptotic performances with $D_m = 0.13$ and $D_M = 2$.*

4.2 Productivity of the resident species

Let us consider the productivity P of the species 1 alone, as the quantity produced per unit of volume and time when the system is at steady steady state. It is given by

$$P := Dx_1^{eq}(D),$$

where $x_1^{eq}(D)$ is the steady state associated with a constant control D (or equivalently $P = \mu_1(s^{eq})(s_{in} - s^{eq})$). The largest value of P is then obtained when the substrate concentration at steady state that maximizes the function $s \mapsto \mu_1(s)(s_{in} - s)$ at a certain $s^\star \in (0, s_{in})$. Then, the maximal productivity is obtained for the dilution rate $D^\dagger = \mu_1(s^\star)$ and one has $x_1^\star = s_{in} - s^\star$. For the chosen value of s_{in}, the maximal productivity of species 1 is obtained for $P_{max} \simeq 0.4289$ (see Fig. 6.7). We have then chosen a threshold x_1^r

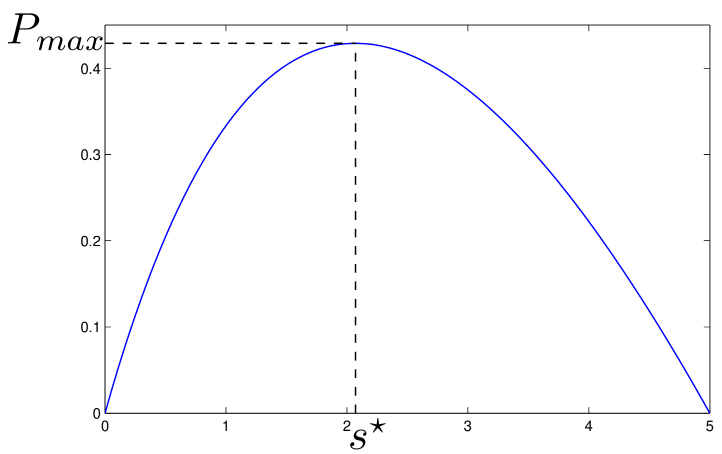

FIGURE 6.7 – *Graph of the function* $s \mapsto \mu_1(s)(s_{in} - s)$ *with* $s_{in} = 5$.

equal to $x_1^\star$ and a value of D_m close to $D^\star$ (such that inequality (6.6) fulfills), so that it does not impact too much the productivity of species 1 as long as species 2 is absent (see Fig. 6.4). In absence of species 2, as the controller takes the value D_m, the productivity of species 1 alone at steady state is $P_m := D_m x_1^m \simeq 0.4216$ (which is quite close to the maximal one).

In presence of species 2, we have considered the average productivity $\bar{P} := \frac{1}{T^\dagger} \int_0^{T_2^\dagger} D(t) x_1^\dagger(t)\, dt$ of species 1 over a period of the asymptotic periodic solution $x^\dagger(\cdot)$, to be compared with the productivity of species 1 alone (see Table 6.1). The percentage of loss of productivity $\frac{P_m - \bar{P}}{P_m}$ is also indicated. It is clear that the presence of the species 2 impacts the productivity of species 1, as there is a new consumer of the resource. However, one can see that it is possible to maintain a relatively low decrease of productivity, under the condition that the parameter ε is not too small.

In conclusion, as already known in the literature in the context of single species (see for instance [BNTM09]), one can appreciate the trade-off between having a high productivity and maintaining a high conversion. The parameter ε is thus a lever of choice for the practitioners.

5 Conclusion

In this work we have proposed a hybrid controller that allows weak resilience in the chemostat model. This controller switches between bounds D_m and D_M, so that the corresponding solution enters infinitely many times into a desirable subset (while each bound is unfavorable to resilience when considered asymptotically). One of the main advantages of this controller is that it does not require the precise knowledge of the growth characteristics of the invader and can cope with a large variety of unknown invading species. In practice, one can simply apply this controller even in absence of invading species, guaranteeing a robustness of the performances against possible future invasions. We have also shown its features in terms of time spent in the desirable subset and productivity thanks to an adequate choice of the bounds D_m and D_M and of the parameter ε, although we do not know if the bang-bang strategy is optimal for the so-called "time of crisis" (see [BR16, BR17, BR19]):

$$\inf_{D(\cdot)} \text{meas} \{t \in [0, T] \; ; \; x(t) \notin \mathcal{K}(x_1^r)\} \, .$$

This amounts to minimize w.r.t. admissible controls the time spent by trajectories outside the desirable set $\mathcal{K}(x_1^r)$ either over a finite given horizon $[0, T]$ (which is more relevant in the present context), or over $[0, +\infty)$ using the notion of finitely optimal control as in [AV15]. This could be the matter of a future work. When practitioners possess more information on the growth functions (such as averages and variabilities, in particular on the possible invasive species), it would be interesting to compare the performances of the proposed controller with more classical approaches such as model predictive control. This could be also studied in a future work.

CHAPITRE 7

CONCLUSIONS ET PERSPECTIVES

Dans la première partie de cette thèse, nous avons étudié plusieurs problèmes de contrôle optimal périodique gouvernés par un système scalaire affine en la commande. La spécificité principale de ces travaux consiste à prendre en compte une contrainte sur la commande de type isopérimétrique (on prescrit la norme 1 du contrôle sur une période de temps donnée, voir les Chapitre 2 et 3). Dans ces travaux, l'apport principal a consisté à déterminer une loi de commande admissible telle que la solution du système associé soit périodique, et pour laquelle la fonction objectif est améliorée par rapport à un contrôle constant. La particularité de ces travaux réside dans le fait que le contrôle constant nominal associé n'est pas nécessairement optimal pour le problème d'optimisation statique (à l'équilibre). Notre approche diffère donc de celles employées par exemple autour de la notion du π-test (voir [BFG73], [BG80]).

Nous avons tout d'abord étudié un problème de contrôle optimal général et introduit des conditions de convexité et de monotonie sur les données du problème (portant sur une fonction qui combine les fonctions dynamique et coût) qui garantissent l'existence de sur-rendement. Nous avons exploité dans ce cadre le Principe du Maximum de Pontryagin afin de caractériser les solutions optimales et périodiques de ce problème de contrôle sous contrainte intégrale sur la commande. La trajectoire optimale est bang-bang avec deux temps de commutation. Notons que sous ces hypothèses de convexité "globales", ces résultats valent pour toute période $T > 0$. Lorsque les hypothèses de convexité ne sont pas satisfaites globalement, nous avons également mis en évidence l'existence de sur-rendement et caractérisé les solutions optimales mais pour des périodes de temps pas trop grandes ($T \leq T_{max}$). Les résultats théoriques obtenus dans [BRTss] ont été ensuite appliqués dans plusieurs situations :

- dans le domaine halieutique, pour un modèle mathématique qui décrit l'évolution de la biomasse d'une espèce exploitée dans un écosystème marin pour un critère bio-économique,
- en biotechnologie, pour étudier un problème d'épuration de l'eau.

En écologie également, on considère le modèle de chémostat qui décrit la croissance d'une espèce de micro-organismes (de concentration x) qui se nourrit sur un substrat (de concentration s). Les résultats théoriques obtenus dans le cas général [BRTss] montrent que les propriétés de la fonction de croissance du micro-organisme (convexité, concavité, croissance) sont fondamentales pour savoir si la performance de conversion de l'écosystème (qualité de l'eau en sortie) peut être améliorée par un contrôle périodique (le taux de dilution) non constant. Dans le Chapitre 3, nous avons aussi considéré un problème "dual" où l'on cherche une loi de commande périodique qui maximise la quantité d'eau à traiter sur une période et pour laquelle la valeur moyenne en substrat doit respecter un seuil (contrainte isopérimétrique sur l'état). Grâce à un résultat de dualité qui relie les fonctions valeurs des deux problèmes, nous avons montré que la solution du problème dual peut être déduite de celle du problème primal.

Dans le Chapitre 4, nous avons montré que la loi de commande périodique (loi bang-bang avec deux commutations) peut être utilisée pour un problème lié à l'identification des taux de croissance. Nous avons proposé un algorithme robuste permettant de distinguer entre deux types de cinétiques. Cet algorithme repose sur le choix du taux de dilution selon deux phases stationnaire et périodique.

Pour les problèmes de sur-rendement, une première perspective serait de trouver des solutions optimales périodiques lorsque les conditions de convexité ne sont pas satisfaites globalement, plus précisément, lorsque la période est supérieure à T_{max}. On s'attend à ce que le contrôle bang-bang ne reste pas optimal pour les périodes qui dépassent T_{max} mais on peut néanmoins garantir les mêmes performances obtenues avec T_{max} en prenant $T = nT_{max}$ et des solutions T_{max}-périodiques. Une deuxième perspective serait d'étudier le cas de plusieurs espèces dans un chemostat. Considérons par exemple le modèle:

$$\begin{cases} \dot{s}(t) & = & D(t)(s_{in} - s) - \mu_1(s)x_1 - \mu_2(s)x_2, \\ \dot{x}_1(t) & = & (\mu_1(s) - D(t))x_1, \\ \dot{x}_2(t) & = & (\mu_2(s) - D(t))x_2, \end{cases} \tag{7.1}$$

où $D : [0, T] \to [D_m, D_M]$ est la variable du contrôle vérifiant la contrainte:

$$\frac{1}{T}\int_0^T D(t)\mathrm{d}t = \bar{D}, \tag{7.2}$$

et les fonctions μ_i sont supposées monotones. Soit $\bar{s} \in (0, s_{in})$ défini comme $\bar{s} := \min_i \lambda_i(\bar{D})$ où $\lambda_i(\bar{D})$ est le seuil de croissance de l'espèce i qui correspond à $D = \bar{D}$ dans le modèle (7.1). La première question qui se pose est de chercher les conditions pour lesquelles une commande D périodique vérifiant (7.2) génère des solutions périodiques de (7.1) avec:

$$\frac{1}{T}\int_0^T s(t)\mathrm{d}t < \bar{s}.$$

Ensuite, une perspective serait de trouver (s'il existe) une loi de commande périodique optimale minimisant la concentration moyenne en substrat. Plusieurs cas sont alors à étudier suivant les graphes des fonctions de croissance ainsi que les valeurs des paramètres D_m, D_M et $\bar{D}$.

Dans un autre contexte, nous avons introduit la notion de « faible résilience » en considérant un problème d'invasion dans un chémostat. Nous avons considéré le cas où une seule espèce (de concentration x_1) qui se nourrit d'un substrat (de concentration s) est présente à l'équilibre dans un milieu réactionnel alimenté à débit constant. Dans un deuxième temps, une nouvelle espèce de micro-organismes apparaît à faible concentration (l'espèce 2 de concentration x_2 dans (7.1)). Lorsque les fonctions de croissance des deux espèces sont alternées (hypothèse 2.2 du Chapitre 5), alors on montre un résultat préliminaire de viabilité qui nous a conduit à introduire ce concept de « faible résilience ». On définit l'ensemble $\mathcal{K}(x_1^r)$ comme

$$\mathcal{K}(x_1^r) := \{x \in \mathbb{R}_+^2 \; ; \; x_1 + x_2 \leq s_{in} \; ; \; x_1 \geq x_1^r \;\text{ and }\; x_2 > 0\},$$

et on montre que sous certaines hypothèses sur les paramètres, le noyau de viabilité de cet ensemble est $Viab(cl\,\mathcal{K}(x_1^r)) = [x_1^r, s_{in}] \times \{0\}$. Ce résultat est important car il montre qu'en présence de l'espèce 2, la croissance de l'espèce 1 est limitée: sa concentration ne peut pas rester supérieure à x_1^r pour tout temps. Pour cela, nous avons proposé une fonction débit $D(\cdot)$ dite faiblement résiliente et qui alterne au cours du temps entre deux valeurs D_m et D_M bien choisies. Les trajectoires associées vérifient vérifient

$$\text{mes}\,\{t \geq 0\ ;\ x(t) \in \mathcal{K}(x_1^r)\} = +\infty$$

et elles rentrent une infinité de fois dans l'ensemble $\mathcal{K}(x_1^r)$ sans jamais pouvoir y rester du fait de la présence de espèce 2. Nous avons montré que toute trajectoire associée à cette commande converge asymptotiquement vers une solution périodique. Nous avons également proposé de réécrire cette fonction $D(\cdot)$ comme un retour d'état *hybride* associé à un automate à trois états possibles. Cette représentation a l'avantage de ne pas exiger une connaissance parfaite des caractéristiques de croissance des espèces. En outre, nous avons étudié numériquement l'impact du paramètre ε définissant loi de commande, sur le temps total passé dans l'ensemble de contraintes désirées ainsi que la productivité moyenne de l'espèce d'intérêt (espèce 1). Nous avons observé que ces deux valeurs sont élevées si ε est très faible.

Comme perspective et pour compléter cette étude, nous souhaiterions étudier et caractériser une commande qui minimise le temps passé à l'extérieur de l'ensemble $\mathcal{K}(x_1^r)$ ce qui revient à considérer le problème de temps de crise suivant:

$$\inf_{u(\cdot)\in\mathcal{U}}\ \text{mes}\,\{t \in [0,T]\ ;\ x_u(t) \notin \mathcal{K}(x_1^r)\}\,,$$

où $\mathcal{U}$ est l'ensemble des contrôles admissibles donné par:

$$\mathcal{U} := \{u : [0,T] \to [u_m, u_M]\,;\, u \text{ mes.}\},$$

et $x_u(\cdot)$ est solution du système (7.1) (avec u au lieu de D) associée à $u \in \mathcal{U}$. Par analogie avec la synthèse précédente, on s'intéressera au cas où les solutions du système sont périodiques.

A terme, on pourrait aussi se poser la question de l'existence de trajectoires faiblement résilientes dans un contexte plus général avec une dynamique

$$\begin{array}{rccc} f: & \mathbb{R}^n \times \mathbb{R}^m & \to & \mathbb{R}^n \\ & (x,u) & \mapsto & f(x,u) \end{array}$$

pour un contrôle $u \in U \subset \mathbb{R}^m$ et un ensemble de contraintes K fixé. Ainsi, on peut distinguer deux cas:

1. lorsque le noyau de viabilité de K (noté $Viab(K)$) est vide, alors toute trajectoire initialisée en un point $x_0 \in K$ quitte nécessairement cet ensemble en temps fini et peut éventuellement y revenir. S'il existe une trajectoire qui re-rentre dans K depuis x_0, on pourrait chercher dans ce cas des lois de commande faiblement résilientes pour K, c'est à dire, on cherche des contrôles u pour lesquels les solutions associées vérifient:
$$\text{mes}\,\{t \geq 0\ ;\ x_u(t) \in K\} = +\infty.$$
On pourrait également étudier dans ce cas le problème de temps de crise pour K.

2. lorsque $Viab(K)$ est non vide alors toute trajectoire initialisée en un point $x_0 \in K \setminus Viab(K)$ quitte nécessairement cet ensemble en temps fini et peut éventuellement y revenir. Pour une condition initiale dans $K \setminus Viab(K)$, si le problème de rejoindre $Viab(K)$ en temps minimal admet une solution, on dit que la trajectoire associée à cette condition initiale est résiliente. Si ce problème n'admet pas de solutions alors il serait intéressant de chercher des contrôles faiblement résilients pour $K \setminus Viab(K)$ (les trajectoires associées rentrent et sortent de cet ensemble une infinité de fois) et de résoudre le problème de rejoindre $K \setminus Viab(K)$ en temps minimal depuis une condition initiale en dehors de K.

ANNEXE A

The aim of these chapter is to give an introduction to the theory of optimal control for finite dimensional systems and in particular to the use of the Pontryagin Maximum Principle for the explicit construction of an optimal synthesis [Ces12], [Vin10], [Cla13], [Bon19].

Control Theory deals with systems that can be controlled, *i.e.* whose evolution is governed by some external agent (controller). This situation is modeled by a control system that can be defined as a system of differential equations depending on $u(t) \in U \subseteq \mathbb{R}^m$, namely:

$$\dot{x}(t) = f(x(t), u(t)), \tag{A.1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. For a given $T > 0$, we consider the set of admissible control functions defined as:

$$\mathcal{U} := \{u : [0,T] \to U \ ; \ u \text{ meas.}\}.$$

Given an initial state $x_0 \in \mathbb{R}^n$, a target $\mathcal{S} \subset \mathbb{R}^n$, a running cost (or Lagrangian) $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, we consider the optimal control problem:

$$\inf_{u \in \mathcal{U}} J(u) := \int_0^T \ell(x(t), u(t))\, \mathrm{d}t, \tag{A.2}$$

for the system (A.1) with initial and terminal condition

$$x(0) = x_0 \ , \ x(T) \in \mathcal{S}. \tag{A.3}$$

The set $\mathcal{S}$ is supposed to be closed and convex. Recall first that for $x \in \mathcal{S}$, the normal cone (in the sense of convex analysis) to $\mathcal{S}$ at point x is defined by:

$$N_{\mathcal{S}}(x) := \{p \in \mathbb{R}^n \ ; \ p \cdot (y - x) \leq 0, \ \forall y \in \mathcal{S}\}.$$

In view of the above problem, several natural questions arise:

1. Existence of a solution $x(\cdot)$ reaching the target $\mathcal{S}$ at the time T, from an initial point x_0.
2. Existence of an optimal solution of (P).
3. Characterization of the optimal synthesis.

Point 1 requires to define a solution of the controlled differential system and the use of the Cauchy-Lipschitz Theorem. Once solutions $x(\cdot)$ have been defined, the question is to know if $\mathcal{S}$ can be connected from x_0 et time T (known as the controllability problem). Point 2 requires to provide standard assumptions about the system so that there is a solution of the optimal control problem, among all admissible controls $u \in \mathcal{U}$. Finally, the Pontryagin Maximum Principle provides the necessary optimality conditions satisfied by an optimal control.

We begin by stating the Filippov Existence Theorem (see [Fil59], [Ces12], [Vin10]).

Theorem 1. *We introduce the following hypotheses on the dynamics:*

(i) The set U is compact.

(ii) The mapping f is continuously differentiable in $\mathbb{R}^n \times U$ and satisfies the linear growth condition: there exists $C > 0$ such that:

$$\mid f(x,u) \mid \leq C(1+ \mid x \mid), \ \forall (x,u) \in \mathbb{R}^n \times U.$$

(iii) The velocity set defined by:

$$F(x) := \{(y,z) \in \mathbb{R}^n \times \mathbb{R} \ , \ \exists u \in U \text{ s.t. } y = f(x,u) \text{ and } z \geq \ell(x,u)\},$$

is a non-empty convex set for any $x \in \mathbb{R}^n$.

(iv) The target $\mathcal{S}$ is attainable from x_0 in time T.

Then, there is an optimal solution of (P)*.*

This result relies on the compactness property of the set of trajectories of (A.1) (see [Cla13] for more details).

The *Pontryagin Maximum Principle* (PMP) is one of the cornerstones of optimal control theory. It gives a set of necessary conditions by which the optimal control may be determined. Before stating this theorem, we present some important definitions related with the optimization problem (P).

Remark 0.1. *The statement of the PMP does not require the former conditions about the existence of optimal solutions but only the regularity conditions on* f *and* ℓ *that we give below.*

Definition 1 (Admissible trajectory)**.** *We say that* (x, u) *is admissible if it satisfies* (A.1)-(A.3) *and the objective function* $J(u)$ *is well defined.*

Definition 2 (Local minimizer)**.** *We say that* $u^\star$ *is a local minimum of* (A.2) *in* L^1 *if and only if there exists* $\varepsilon > 0$ *such that for any admissible control* $u \in \mathcal{U}$*, one has*

$$\|u - u^\star\|_{L^1} \leq \varepsilon \Rightarrow J(u) \geq J(\bar{u}).$$

We define the *Hamiltonian* $H : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ by:

$$H(x, p, p_0, u) := p \cdot f(x, u) + p_0 \ell(x, u),$$

and impose the following hypotheses on the dynamics and cost function:

(H1) The set U is a non-empty closed subset of $\mathbb{R}^m$,

(H2) The mappings f and ℓ are C^1 with Lipschitz continuous derivatives w.r.t. (x, u) on every compact subset.

One then has the following result.

Theorem 2 (PMP)**.** *Suppose that (H1)-(H2) hold true. Let* $u^\star$ *be a local minimum of* (A.2) *and* $x^\star(\cdot)$ *be the associated solution of* (A.1) *with* $x^\star(0) = x_0$*. Then there exists an absolutely continuous map* $p : [0, T] \to \mathbb{R}^n$ *and* $(p_0, \alpha) \in \mathbb{R}_- \times N_{\mathcal{S}}(x(T))$ *such that* $(p(\cdot), p_0) \neq 0$ *and for almost every* $t \in [0, T]$*:*

$$\dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), p(t), p_0, u(t)), \tag{A.4}$$

with the transversality condition:

$$p(T) = \alpha. \tag{A.5}$$

Moreover, the following Hamiltonian condition is satisfied for almost every $t \in [0, T]$:

$$u^\star(t) \in \arg\min_{\omega \in U} H(x(t), p(t), p_0, \omega). \tag{A.6}$$

Note that we can also deal with a general closed subset $\mathcal{S}$ that is not necessarily convex. In that case, the transversality condition writes $p(T) = \alpha \in N_{\mathcal{S}}(x(T))$ where $N_{\mathcal{S}}(x)$ stands for the proximal normal cone to $\mathcal{S}$ at x.

Let us now state the following definitions.

Definition 3 (Extremal)**.** *We say that $(x(\cdot), p(\cdot), p_0, u(\cdot))$ is an extremal of the problem* (A.2) *if it satisfies* (A.1)-(A.4)-(A.5)-(A.6)*. The corresponding trajectory $x(\cdot)$ is called extremal trajectory.*

Definition 4 (Normal and abnormal extremal)**.** *An extremal is said to be normal if $p_0 < 0$ whereas, if $p_0 = 0$, we say that the extremal is abnormal.*

Definition 5 (Singular arc)**.** *Let $(x(\cdot), p(\cdot), p_0, u(\cdot))$ be an extremal of* (A.2)*. We say that the control is singular on a time interval $[t_1, t_2]$ if the condition:*

$$\frac{\partial H}{\partial u}(x(\cdot), p(\cdot), p_0, u(\cdot)) = 0, \text{ a.e. } t \in [t_1, t_2],$$

is satisfied. The corresponding portion of trajectory is called singular trajectory or singular arc.

Let us note that for autonomous problems (*i.e* when the functions f and ℓ are independent of time), that is the case here, the Hamiltonian is conserved along any extremal trajectory which means that the function

$$t \mapsto H(x(\cdot), p(\cdot), p_0, u(\cdot))$$

is constant for almost every $t \in [0, T]$. The constancy of the Hamiltonian is a useful property in many optimal control problems. It can be used to determine the switching curves (see the next paragraph) in optimal control problems governed by a one-dimensional control system.

Particular cases An interesting application of the PMP is in the special case of a scalar control u taking values in the bounded set $U = [\underline{u}, \bar{u}]$ (*i.e* $m = 1$) and the affine control system

$$\dot{x}(t) = f(x) + u(t)g(x), \tag{A.7}$$

with $f, g : \mathbb{R}^n \to \mathbb{R}^n$. The control problem that we consider here is to find a control that steers a given initial point $x_0 \in \mathbb{R}^n$ into a target set $\mathcal{S} \subset \mathbb{R}^n$ in minimum time. It consists then to take $\ell(x) = 1$.
The Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is linear w.r.t u and writes:

$$H(x, p, p_0, u) := p \cdot f(x) + up \cdot g(x) + p_0.$$

To express the optimal control law in a convenient form, let us define the *switching function* associated to the control u as:

$$\phi(t) := \frac{\partial H}{\partial u}(x(t), p(t), p_0, u(t)) = p(t) \cdot g(x(t)),\ t \in [0, T].$$

Thus, the Hamiltonian condition implies the following control law:

$$\begin{cases} u(t) = \bar{u} & \text{if} \quad \phi(t) > 0, \\ u(t) = \underline{u} & \text{if} \quad \phi(t) < 0, \end{cases}$$

for almost every $t \in [0, T]$. We say that the control is *bang-bang* if $t \mapsto u(t)$ is piecewise constant control function taking alternatively extremal values of the set U with a finite number $p \in \mathbb{N}^\star$ of switching times. If the control u is singular over a time interval $[t_1, t_2]$ then one has

$$\phi(t) = 0,\ \forall t \in [t_1, t_2].$$

Pontryagin's Principle does not yield any information directly on singular controls. A way to obtain further informations on singular controls is to differentiate the switching function $\phi(\cdot)$ w.r.t. the time. We say that a singular arc is of order 1 if u does not appear explicitly in the expression of $\dot{\phi}$ and $\ddot{\phi}$ depends linearly on the control u. A direct computation of these derivatives yields

$$\dot{\phi}(t) = p(t) \cdot [f, g](x(t)),\ t \in [0, T],$$

$$\ddot{\phi}(t) := p(t) \cdot [f,[f,g]](x(t)) + u(t)p(t) \cdot [g,[f,g]](x(t)),\ t \in [0,T],$$

where $[f,g](x)$ is the *Lie bracket* of the vectors f and g for x, defined on $\mathbb{R}^n$ by

$$[f,g](x) := Dg(x)f(x) - Df(x)g(x).$$

We provide in the following result necessary conditions known as Goh conditions and the Legendre-Clebsch condition for obtaining further informations on singular controls (of first order) [SL12], [BJ75].

Theorem 3. *Let $(x(\cdot), p(\cdot), u(\cdot))$ be a normal extremal defined over $[0,T]$. Suppose that it is singular over a time interval $[t_1,t_2]$ and that the corresponding singular control is admissible (i.e. $u(t) \in (\underline{u}, \bar{u})$ for almost every $t \in [t_1,t_2]$). Then, the following Goh conditions are satisfied for every $t \in [t_1,t_2]$*

$$\begin{cases} \phi(t) = 0, \\ \dot{\phi}(t) = 0. \end{cases}$$

If the singular extremal is optimal, the Legendre-Clebsch condition given as

$$p(t) \cdot [g,[f,g]](x(t)) \leq 0,$$

holds true over $[t_1,t_2]$.

Note that the Legendre-Clebsch condition is similar to the Legendre condition in the calculus of variations that is a necessary second-order condition for local optimality.

In Chapter 2 (Proposition 3.1), we have given an optimal synthesis for the control problem

$$\inf_{u\in\mathcal{U}} \frac{1}{T}\int_0^T \ell(x(t))\,\mathrm{d}t$$

with the dynamics

$$\begin{cases} \dot{x} &= f(x) + ug(x), \\ \dot{y} &= u, \end{cases}$$

and the boundary condition

$$(x(0), y(0)) = (\bar{x}, 0) \quad , \quad (x(T), y(T)) = (\bar{x}, \bar{u}T).$$

We have shown that an optimal trajectory does not have a singular arc using convexity and monotonicity assumptions (hypotheses (H1)-(H3)). Here, we present another proof based on Legendre-Clebsch's condition. We denote by λ the adjoint vector as introduced in Chapter 2, instead of p. Without any loss of generality, we take $\lambda_0 = -1$ and assume that hypotheses (H1)-(H3) hold true. A simple computation of ϕ and $\dot{\phi}$ yield

$$\phi(t) = \lambda_x(t)g(x(t)) + \lambda_y,$$

$$\dot{\phi}(t) = \lambda_x(t)[f(x(t))g'(x(t)) - f'(x(t))g(x(t))] + \ell'(x(t))g(x(t)).$$

Recall that $\psi = -f/g$ therefore one obtains

$$\dot{\phi}(t) = \lambda_x(t)g^2(x(t))\psi'(x(t)) + \ell'(x(t))g(x(t)).$$

If there exists singular extremal over a time interval $[t_1, t_2]$ then one gets according to Theorem 3 stated before

$$\begin{cases} \phi(t) = 0, \\ \dot{\phi}(t) = 0, \end{cases}$$

for every $t \in [t_1, t_2]$. Therefore one can write

$$\dot{\phi}(t) = g(x(t))(\ell'(x(t)) - \lambda_y\psi'(x(t))), \quad \forall t \in [t_1, t_2].$$

A simple computation of the second derivative of ϕ yields

$$\ddot{\phi}(t) = g'(x(t))(\ell'(x(t)) - \lambda_y\psi'(x(t)))\dot{x}(t) + g(x(t))(\ell''(x(t)) - \lambda_y\psi''(x(t)))\dot{x}(t),$$

$t \in [t_1, t_2]$. Consequently, one get

$$\lambda(t) \cdot [g, [f, g]](x(t)) = g^2(x(t))(\ell''(x(t)) - \lambda_y\psi''(x(t))), \quad \forall t \in [t_1, t_2]. \tag{A.8}$$

Using the relation between ψ and γ given as $\psi = \gamma \circ \ell$, then the first and the second derivatives of ψ w.r.t. x write

$$\psi'(x) = (\ell'(x))(\gamma' \circ \ell)(x)\ ,\ \psi''(x) = (\ell'(x))^2(\gamma'' \circ \ell)(x) + \ell''(x)(\gamma' \circ \ell)(x).$$

If we replace this expressions in (A.8), we get

$$\lambda(t) \cdot [g, [f, g]](x(t)) = \frac{(g(x(t))^2)(\ell'(x(t)))^3}{\psi'(x(t))}(\gamma'' \circ \ell)(x(t)), \quad \forall t \in [t_1, t_2].$$

Since γ is strictly convex increasing over $\ell(I)$ and ℓ is increasing over I (hypothesis (H3)) then one obtains

$$\lambda(t) \cdot [g, [f, g]](x(t)) > 0, \quad \forall t \in [t_1, t_2]$$

which contradicts the Legendre-Clebsch condition given in Theorem 3 of this Annex. This allow to exclude singular arcs for optimality.

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