# Periodic solution of switched slow-fast systems

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- Switched system is an hybrid system that can be seen as a continuous system with switching events.
- The switching events are subjected to a switching signal depending on the time, the state or both.
- Switched Slow-Fast System is a system with slow and fast dynamics with a fast switching signal depending on the time.

# Slow-Fast Switched System 2/3

- Slow-fast switched system :

$$\begin{cases} \epsilon^2 \frac{dx}{dt} = f_u(x, y, \epsilon) \\ \frac{dy}{dt} = g_u(x, y, \epsilon), \end{cases} \quad u \in U := \{0, 1\}$$

$$(x, y) \in \mathbb{R}^n \times \mathbb{R}^m, \qquad \epsilon \approx 0$$

- The switching signal $u : \mathbb{R}^+ \to U$ is a periodic piecewise constante function.

# Slow-Fast Switched System 3/3

- To define $u$, we suppose given
  - a continuous 1-periodic real function $\Theta : \mathbb{R} \to [0,1]$,
  - a positive integer $N \in \mathbb{N}^*$,
  
  We define a $P$-periodic switching signal $u$ over one period $P := \epsilon N$ (fast switching) by

$$u(t) = \begin{cases} 0 & \text{if} \quad t \in [\epsilon i, \epsilon(i + \theta_i)), \\ 1 & \text{if} \quad t \in [\epsilon(i + \theta_i), \epsilon(i + 1)), \end{cases} \tag{1}$$

for $i = 0..N-1$ and $\theta_i = \Theta(\frac{i}{N})$.

# Slow-Fast Switched System 3/3

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  for $i = 0..N - 1$ and $\theta_i = \Theta(\frac{i}{N})$.
- Here the switching signal depends only on the time so that the Switched System becomes singularly perturbed and $P$-periodic on the time $t$.

# Slow-fast systems 1/6

We consider two scales of the time, the slow time $t$ and the fast time $\tau = t/\epsilon^2$, the switched system has two limit systems when $\epsilon$ tends to 0,

- at the slow time $t$ :

$$\left\{ \begin{array}{rcll} \epsilon^2 \frac{dx}{dt} & = & f_u(x,y,\epsilon), & x(0) = x_0 \\ \frac{dy}{dt} & = & g_u(x,y,\epsilon), & y(0) = y_0 \end{array} \right. \longrightarrow \left\{ \begin{array}{rl} 0 & = f_u(x,y,0) \\ \frac{dy}{dt} & = g_u(x,y,0) \end{array} \right.$$

- at the fast time $\tau$ :

$$\left\{ \begin{array}{rcll} \frac{dx}{d\tau} & = & f_u(x,y,\epsilon), & x(0) = x_0, \\ \frac{dy}{d\tau} & = & \epsilon^2 g_u(x,y,\epsilon), & y(0) = y_0, \end{array} \right. \longrightarrow \left\{ \begin{array}{rcl} \frac{dx}{d\tau} & = & f_u(x,y,0) \\ \frac{dy}{d\tau} & = & 0 \end{array} \right.$$

# Slow-fast systems 2/6

- $x$ varies quickly and is approximated by the solution of the *boundary layer equation*

$$\frac{dx}{d\tau} = f_u(x, y_0, 0), \quad x(0) = x_0$$

- The equation $\frac{dx}{d\tau} = f_u(x, y, 0)$ is the *fast equation*.
- Assume that the solution of the *boundary layer equation* tends to a stationary point $\xi_u(y_0)$, solution of $f_u(x, y_0, 0) = 0$.
- the set of the stationary points of the *fast equation* is the *slow manifold*. (set of the roots of the equation $f_u(x, y, 0) = 0$).

# Slow-fast systems 3/6

The solution of the switched system is defined for all $\tau \geq 0$ and tends to a stationary point $(\xi_u(y_0), y_0)$. Hence a fast transition brings the solution of system near the slow manifold where a slow motion takes place and is approximated by the solution of the *reduced problem*

$$\frac{dy}{dt} = g_u(\xi_u(y), y, 0), \quad y(0) = y_0$$

which is the limit system at the slow time $t$:

$$\begin{cases} 0 = & f_u(x, y, 0), \\ \frac{dy}{dt} = & g_u(x, y, 0), \qquad y(0) = y_0 \end{cases}$$

# Slow-fast systems 4/6

The above description is precised by Tykhonov's theorem,

- **H1.** For each $u \in \{0, 1\}$, $f_u$ and $g_u$ are $C^1$-functions.
- **H2.** For each $u \in \{0, 1\}$, there exists an $m$-dimensional compact manifold $\mathcal{L}_u$, subset of the slow manifold, given as a graph of a continuous function $\xi_u$ defined in a compact domain $K_u \subset \mathbb{R}^m$ and for all $y \in K_u$, $\xi_u(y)$ is an isolated root of $f_u(x, y, 0) = 0$.
- **H3.** For each $u \in \{0, 1\}$ and each $y \in K_u$, the point $x = \xi_u(y)$ is an exponentially asymptotically stable equilibrium point of the fast equation.

# Slow-fast systems 5/6

## Remark

- **H3** *means that, by using implicite function theorem, one can define at least locally the slow manifold as a graph of a continuous function.*
- **H2** *says that one can extend globally this definition of* $\mathcal{L}_u$ *over a compact domain* $K_u$.

# Slow-fast systems 6/6

## Theorem

*Under hypothesis H1-H3, let*

- $(x_0, y_0)$ *in the basin of attraction of* $\mathcal{L}_u$*,*
- $\varphi(\tau)$ *be the solution of the boundary layer equation,*
- $\psi(t)$ *be the solution of the reduced problem,*
- $I$ *the positive interval of definition of* $\psi$*.*

*Let* $T \in I$ *then*
$\forall \eta > 0$, $\exists \epsilon_0$, $\forall \epsilon \in (0, \epsilon_0)$, *the solution* $(x(t), y(t))$ *of switched system starting at* $(x_0, y_0)$ *is defined at least on* $[0, T]$ *and* $\exists \Pi > 0$, $\epsilon \Pi < \eta$ *and*

$$\left\{ \begin{array}{lll} \|x(\epsilon^2\tau) - \varphi(\tau)\| < \eta, & \text{for} & 0 \leq \tau \leq \Pi, \\ \|x(t) - \xi(\psi(t))\| < \eta, & \text{for} & \epsilon^2\Pi \leq t \leq T, \\ \|y(t) - \psi(t)\| < \eta, & \text{for} & 0 \leq t \leq T. \end{array} \right.$$

# Main Results 1/3

- **H4.** The slow manifolds $\mathcal{L}_u$ for $u \in \{0, 1\}$ of the switched system have the same basin of attraction $\mathcal{B}(\mathcal{L}_u)$.
- **H4** allows us to define the slow manifolds $\mathcal{L}_u$ on the same compact domain $K$

$$\mathcal{L}_u = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \,/ x = \xi_u(y),\ \forall y \in K\}$$

- Let $N \in \mathbb{N}^*$ fixed and $\bar{\Theta} = \sum_{i=0}^{N-1} \theta_i$.
  We define the differential equation

$$\frac{d\bar{y}}{dt} = G(\bar{y}) := \bar{\Theta} g_0(\xi_0(\bar{y}), \bar{y}, 0) + (N - \bar{\Theta}) g_1(\xi_1(\bar{y}), \bar{y}, 0)$$

# Main Results 2/3

- Theorem

  *If there exists an exponentially asymptotically stable equilibrium point $\bar{y}_0 \in \mathbb{R}^m$ of equation*

$$(*) \quad \frac{d\bar{y}}{dt} = G(\bar{y}) := \bar{\Theta} g_0(\xi_0(\bar{y}), \bar{y}, 0) + (N - \bar{\Theta}) g_1(\xi_1(\bar{y}), \bar{y}, 0)$$

  *then $\forall \eta > 0$, $\exists \epsilon_0$ such that $\forall \epsilon < \epsilon_0$, the switched system admits a $P$-periodic solution starting at $(x^*, y^*)$ such that*

$$\begin{cases} \|x^* - \xi_1(\bar{y}_0)\| < \eta, \\ \|y^* - \bar{y}_0\| < \eta. \end{cases}$$

- Theorem

  *If there exists an asymptotically orbitaly stable $T$-periodic solution of equation (*) starting at $\bar{y}_0 \in \mathbb{R}^m$, then $\forall \eta > 0$, $\exists \epsilon_0$ such that $\forall \epsilon < \epsilon_0$ satisfying $\frac{T}{\epsilon N} = L \in \mathbb{N}$, the switched system admits a $T$-periodic solution starting at $(x^*, y^*)$ such that*

$$\begin{cases} \|x^* - \xi_1(\bar{y}_0)\| < \eta, \\ \|y^* - \bar{y}_0\| < \eta. \end{cases}$$

# Main Results 3/3

- We define the differential equation

$$(**) \qquad \frac{d\bar{y}}{dt} = G(t, \bar{y})$$

where

$$G(t, \bar{y}) = \Theta(t) g_0(\xi_0(\bar{y}), \bar{y}, 0) + (1 - \Theta(t)) g_1(\xi_1(\bar{y}), \bar{y}, 0).$$

# Main Results 3/3

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- Theorem

*If there exists an asymptotically stable* $1$*-periodic solution of averaged equation (**) starting at the point* $\bar{y}_0 \in \mathbb{R}^m$*, then* $\forall \eta > 0$*,* $\exists \epsilon_0$ *such that* $\forall \epsilon < \epsilon_0$*, the switched system admits a* $P$*-periodic solution starting at* $(x^*, y^*)$ *such that*

$$\begin{cases} \|x^* - \xi_1(\bar{y}_0)\| < \eta, \\ \|y^* - \bar{y}_0\| < \eta. \end{cases}$$

# Exemple 1

- The dynamic of a population of microalgual in a fluctuating environment (J.-M. Guarini, G. Blanchard, P. Gros)

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- In the positive 1/4-plane $\{(x,y) \in \mathbb{R}^2 / \ x \geq 0, y \geq 0\}$.
-

$$\begin{cases} \epsilon^2 \frac{dx}{dt} = f_0(x,y,\epsilon) := & ry(1-x) + \epsilon x[(p-r)(1-x) - \epsilon\mu] \\ \frac{dy}{dt} = g_0(x,y,\epsilon) := & px - \mu y, \end{cases}$$

$$\begin{cases} \epsilon^2 \frac{dx}{dt} = f_1(x,y,\epsilon) := & -(1+\epsilon^2\mu)x \\ \frac{dy}{dt} = g_1(x,y,\epsilon) := & -\mu y. \end{cases}$$

$u$ is a $P$-periodic switching signal defined over one period $P = \epsilon N$ by

$$u(t) = \begin{cases} 0 & \text{if} \quad t \in [\epsilon i, \epsilon(i+\theta_i)), \\ 1 & \text{if} \quad t \in [\epsilon(i+\theta_i), \epsilon(i+1)), \end{cases} \tag{2}$$

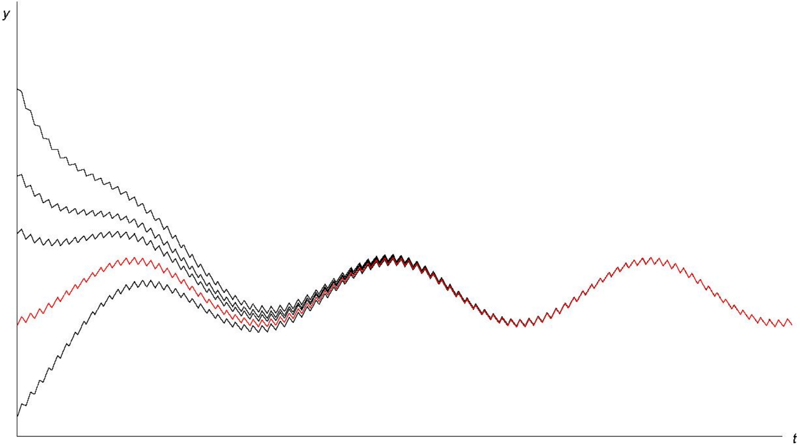
where $i = 0..N-1$ and $0 \leq \theta_i \leq 1$.

- Slow manifold $\mathcal{L}_1$ is the straight line $x = 0$.
- $N = 1$, the first theorem implies existence of an $\epsilon$-periodic solution close to $(1, \bar{y})$.
- Averaged equation

$$\frac{dy}{dt} = G(y) := -\mu y + p\theta_0$$

which admits the equilibrium point $\bar{y} = \frac{p\theta_0}{\mu}$.

- $\theta(t) = (2 + \sin(2\pi t))/4$, the averaged equation (**) becomes
$$\frac{dy}{dt} = G(t, y) := -\mu y + p\theta(t)$$
which admits an asymptotically stable periodic solution
$$y(t) = \frac{p}{4}\left(\frac{2}{\mu} + \frac{1}{2\pi + \mu}\sin(2\pi t)\right)$$
- If $N = O(1/\epsilon)$ we can apply the third theorem.


y
t

# Example 2

Consider in $\mathbb{R}^3$ the following switched system

$$\begin{cases} \epsilon^2 \frac{dx}{dt} = f_0(x,u,v,\epsilon) := & -x - u^2 + 1, \\ \frac{du}{dt} = g_0(x,u,v) := & v, \\ \frac{dv}{dt} = h_0(x,u,v) := & -u + \gamma x v, \end{cases} \tag{3}$$

and

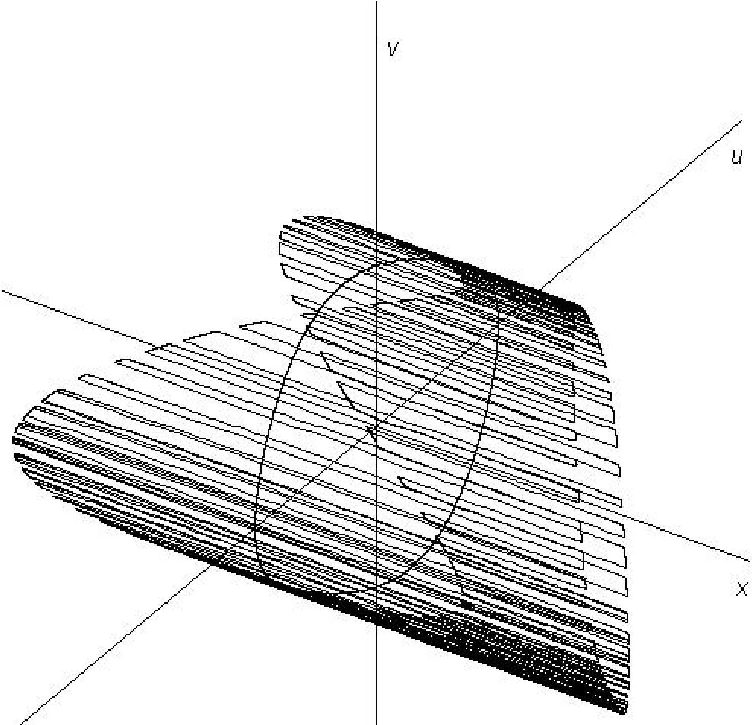
$$\begin{cases} \epsilon^2 \frac{dx}{dt} = f_1(x,u,v) := & -x + u + \beta, \\ \frac{du}{dt} = g_1(x,u,v) := & v, \\ \frac{dv}{dt} = h_1(x,u,v) := & \beta - x + \gamma(x + 1 - \beta)(1 - u)v, \end{cases} \tag{4}$$

$\gamma > 0$, $\beta > 2$, $\Theta(t) = (2 + \sin(2\pi t))/4$

- Slow manifolds
  $\mathcal{L}_0 = \{(x,u,v) \in \mathbb{R}^3, x = \xi_0(u,v) := 1 - u^2\}$ and
  $\mathcal{L}_1 = \{(x,u,v) \in \mathbb{R}^3, x = \xi_1(u,v) := u + \beta\}$
- The reduced corresponding equation

$$\left\{ \begin{array}{rcl} \frac{du}{dt} &=& v, \qquad i = 0, 1, \\ \frac{dv}{dt} &=& -u + \gamma(1 - u^2)v, \end{array} \right. \tag{5}$$

that is van der Pol equation which admits, since $\gamma > 0$, an asymptotically orbitally stable periodic solution. So we can apply the second theorem.


v
u
x

# Averaging 1/2

- Suppose $N = O(1)$ and let $(x(t), y(t))$ the solution of the switched system starting at time $t_0 = 0$ from $(x_0, y_0) \in \mathcal{L}_1$ (*i.e.* $x_0 = \xi_1(y_0)$) and consider the sequence $(x_n, y_n),\ n \in \mathbb{N}$, defined

$$(x_n, y_n) = (x(t_n), y(t_n)), \text{ with } t_n = nP$$

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- **Proposition**

*For each $\eta > 0$ there exists $\epsilon_0$ such that for all $\epsilon < \epsilon_0$ the sequence $(x_n, y_n)$ verifies*

$$\begin{cases} \|x_n - \xi_1(y_n)\| < \eta, \\ \|y_n - \bar{y}(t_n)\| < \eta. \end{cases}$$

*where $\bar{y}(t)$ is the solution of*

$$\frac{d\bar{y}}{dt} = G(\bar{y}) := \bar{\Theta} g_0(\xi_0(\bar{y}), \bar{y}, 0) + (N - \bar{\Theta}) g_1(\xi_1(\bar{y}), \bar{y}, 0)$$

# Averaging 2/2

- Suppose now $N = O(1/\epsilon)$ and consider the sequence $(x_n, y_n),\ n \in \mathbb{N}$, defined by

$$(x_n, y_n) = (x(t_n), y(t_n)),\ \text{with } t_n = n\epsilon$$

# Averaging 2/2

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*where $\bar{y}(t)$ is the solution*

$$\frac{d\bar{y}}{dt} = G(t, \bar{y}) = \Theta(t) g_0(\xi_0(\bar{y}), \bar{y}, 0) + (1 - \Theta(t)) g_1(\xi_1(\bar{y}), \bar{y}, 0).$$

*starting at $t_0 = 0$ from the point $y_0$.*